MATH 280C SPRING 2016 HOMEWORK 2

Due date: Friday, May 27, 2016

Rules: Write as efficiently as possible – and think carefully what to write and what not. Each problem indicates maximum allowed length; this is usually much more than needed. If you type, do not use font smaller than 10pt. I will not grade any text that exceeds the specified length.

1. (5 lines) Let σ be a Π_1-sentence in the language of Arithmetic. Recall that Q is Robinson Arithmetic.

Prove that if $\mathsf{Q} \cup \{\sigma\}$ is consistent then

$$\mathfrak{N} \models \sigma$$

where $\mathfrak{N}$ is the standard model of Arithmetic.

2. (2/3 page) Recall that $\mathcal{N}$ is the Baire space. Prove that there is an open set $G \subseteq \mathcal{N} \times \mathcal{N}$ such that G is universal, that is, for every open set $A \subseteq \mathcal{N}$ there is some $e \in \mathcal{N}$ such that

$$A = \{a \in \mathcal{N} \mid (e, a) \in G\}$$

Notice that in a similar way one can construct a universal open set $G \subseteq \mathcal{N} \times X$ whenever X is a Polish space.

Hint. $\mathcal{N}$ has countable basis of topology.

3. (2/3 page) Recall the Borel Hierarchy $(\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha \mid \alpha < \omega_1)$. Work in $\mathcal{N}$. By induction on α prove the following.

(a) There is a $\mathbf{\Pi}^0_1$-set which is not $\mathbf{\Sigma}^0_1$.

(b) $\mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha \subseteq \mathbf{\Sigma}^0_{\alpha+1}, \mathbf{\Pi}^0_{\alpha+1}$ whenever $\alpha < \omega_1$.

(c) $\mathbf{\Sigma}^0_\alpha = \mathbf{\Pi}^0_\alpha$ whenever $\alpha \leq \omega_1$ is limit.

4. (3/2 page) Let X be a Polish space. Give topological proofs of the following.

(a) If $B \subseteq X \times \mathcal{N}$ is such that $B = p[C]$ where $C \subseteq X \times \mathcal{N} \times \mathcal{N}$ is closed and $A \subseteq X$ is such that $A = p[B]$ then there is a closed set $C' \subseteq X \times \mathcal{N}$ such that $A = p[C']$.

(b) If $A_n \subseteq X$ is such that $A_n = p[C_n]$ where $C_n \subseteq X \times \mathcal{N}$ is closed for all $n \in \omega$ then there is a closed set $C \subseteq X \times \mathcal{N}$ such that

$$\bigcup_{n \in \omega} A_n = p[C]$$

(c) If $A_n \subseteq X$ is such that $A_n = p[C_n]$ where $C_n \subseteq X \times \mathcal{N}$ is closed for all $n \in \omega$ then there is a closed set $C \subseteq X \times \mathcal{N}$ such that

$$\bigcap_{n \in \omega} A_n = p[C]$$

(d) Conclude that every $\mathbf{\Sigma}^1_1$-set is of the form $p[C]$ where $C \subseteq X \times \mathcal{N}$ is a closed set. Use (a) – (c).

5. Let $X = \mathcal{N}$. Prove (a)–(d) in Problem 4 for this choice of X using tree representations. Formulate the corresponding statements carefully.