MATH 280C SPRING 2016 HOMEWORK 1

Due date: Friday April 22, 2016

Rules: Write as efficiently as possible – and think carefully what to write and what not. Each problem indicates maximum allowed length; this is usually much more than needed. If you type, do not use font smaller than 10pt. I will not grade any text that exceeds the specified length.

1. (1/2 page) Assume $A \subseteq \mathbb{N}$ is such that both A and $\mathbb{N} \setminus A$ are recursively enumerable. Prove that A is recursive.

Hint. Use the characterization of partial recursive functions.

2. (4 x 1/4 page) This exercise concerns the fact mentioned in the lecture that recursively enumerable sets are precisely the ranges of total recursive functions. Prove the following.

(a) If $f : \mathbb{N} \to \mathbb{N}$ is total recursive then $\mathrm{rng}(f)$ is recursively enumerable.
(b) If $A \subseteq \mathbb{N}$ is recursively enumerable then there is a total recursive $f : \mathbb{N} \to \mathbb{N}$ such that $A = \mathrm{rng}(f)$.
(c) If $A \subseteq \mathbb{N}$ is recursively enumerable then there is a total recursive injective $f : \mathbb{N} \to \mathbb{N}$ such that $A = \mathrm{rng}(f)$.
(d) If $f : \mathbb{N} \to \mathbb{N}$ is total recursive and increasing (not necessarily strictly) then $\mathrm{rng}(f)$ is a recursive subset of $\mathbb{N}$.

Hint. Use the characterization of partial recursive functions.

3. (3 x 1/3 page) Prove the following.

(a) If $e \in \mathbb{N}$ then there are infinitely many numbers $e' \in \mathbb{N}$ such that e, e' are Gödel numbers of the same partial recursive function.
(b) If $f : \mathbb{N} \to \mathbb{N}$ is a total recursive function then there are infinitely many numbers e such that $e, f(e)$ are Gödel numbers of the same recursively enumerable function.
(c) If $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a total recursive function then there are infinitely many numbers e such that e is a Gödel number of the recursive function $x \mapsto f(e, x)$.

Hint. Use the *s*-*m*-*n*-theorem and for (b),(c) imitate the proof of the Recursion theorem.

4. (1/2 page) Prove that

$$A = \{e \in \mathbb{N} \mid W_e \neq \varnothing\}$$

is a complete Σ_1-definable set.

Hint. Use the *s*-*m*-*n*-theorem.

5. (2/3 page) Given sets $A, B \subseteq \mathbb{N}$ define

$$A \oplus B = \{2k \mid k \in A\} \cup \{2k + 1 \mid k \in B\}.$$

We say that $A \oplus B$ is the **join** of sets A, B.

(a) Prove that if A is recursively enumerable and $A \leq_m \mathbb{N} \setminus A$ then A is recursive.

(b) Prove that the assumption on A being recursively enumerable in (a) cannot be omitted.

Hint. For (b), consider the join of A and its complement.