MATRIX MULTIPLICATION

We recall the matrix multiplication for matrices of type $n \times n$, that is, for matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

If $A = (a_{ij})$ and $B = (b_{ij})$ are two such matrices, then their product $A \cdot B$ (we often write briefly AB) is a matrix $C = (c_{ij})$ of type $n \times n$ such that

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

This can be visualized as follows

$$\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{a_{i1}} & \cdots & \mathbf{a_{ij}} & \cdots & \mathbf{a_{in}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & \cdots & \mathbf{b_{1j}} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{i1} & \cdots & \mathbf{b_{ij}} & \cdots & b_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{n1} & \cdots & \mathbf{b_{nj}} & \cdots & b_{nn} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & \cdots & \cdots & c_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \cdots & \cdots & \mathbf{c_{ij}} & \cdots & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{n1} & \cdots & \cdots & \cdots & c_{nn} \end{pmatrix}$$

In particular, for matrices of type 2×2 we have

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

The above rule for multiplication applies also to situations where A is of type $m \times n$ and B is of type $n \times p$, that is,

The number of **columns** in A = The number of **rows** in B.

The resulting product $A \cdot B$ is then of the form $m \times p$. Thus, if A is a matrix of type $n \times n$ and $\mathbf{x}$ is a vector of length n, then $\mathbf{x}$ is considered a matrix of type $n \times 1$. Then

$$A\mathbf{x} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{pmatrix}$$