SAMPLE SOLUTIONS

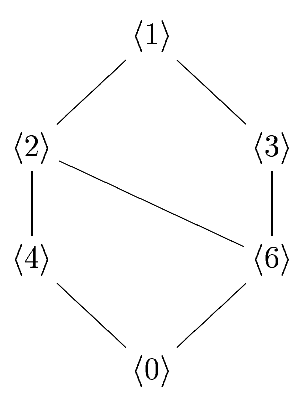
Item 3(a). (5pt) Find all subgroups of $\mathbb{Z}_{12}$ and draw their subgroup diagram.

Sample solution. According to the theorem on subgroups of cyclic groups from the lecture, we know the following: If $G = \langle a \rangle$ is a cyclic group of order s then the subgroups of G are precisely all groups of the form $\langle a^d \rangle$ where d is a divisor of s.

In our case, $\mathbb{Z}_{12} = \langle 1 \rangle$ and has order 12. The group operation is $+_{12}$, so computing in the group $\mathbb{Z}_{12}$ yields: $1^d = d$ for $0 \leq d < 12$ and $1^{12} = 0$. Thus, the subgroups of $\mathbb{Z}_{12}$ are precisely all groups of the form $\langle d \rangle$ where d is a divisor of 12. So d can attain one of the following values: $0, 1, 2, 3, 4, 6$. So the subgroups are:

$$
\begin{aligned}
\langle 1 \rangle &= \{0,1,2,3,4,5,6,7,8,9,10,11\} = \mathbb{Z}_{12} \\
\langle 2 \rangle &= \{0,2,4,6,8,10\} \\
\langle 3 \rangle &= \{0,3,6,9\} \\
\langle 4 \rangle &= \{0,4,8\} \\
\langle 6 \rangle &= \{0,6\} \\
\langle 0 \rangle &= \{0\}
\end{aligned}
$$

For d, d' as above we have $\langle d \rangle \subseteq \langle d' \rangle$ if and only if d' is a divisor of d, so we have the following subgroup diagram:



Item 3(b). (5pt) Find the subgroup of $\mathbb{Z}_{12}$ generated by the set $\{4, 6\}$.

Sample solution. Let H denote the subgroup generated by $\{4, 6\}$. According to the theorem on subgroups of cyclic groups, H is cyclic, so $H = \langle d \rangle$ for some $d \in \mathbb{Z}_{12}$.

For any number $d' \in \mathbb{Z}_{12}$ we have: $4 \in \langle d' \rangle$, if and only if $\langle 4 \rangle \subseteq \langle d' \rangle$, and this holds if and only if d' is a divisor of 4. Similarly, $6 \in \langle d' \rangle$, if and only if $\langle 6 \rangle \subseteq \langle d' \rangle$, and this holds if and only if d' is a divisor of 6. So we conclude that $\{4, 6\} \subseteq \langle d' \rangle$ if and only if d' is a common divisor of 4 and 6. In particular, d is a common divisor of 4 and 6.

We recall that $\langle d_1 \rangle \subseteq \langle d_2 \rangle$ if and only if d_2 is a divisor of d_1. Now H is defined to be the **smallest** subgroup of $\mathbb{Z}_{12}$ that contains $\{4, 6\}$, so d must be the **largest** common divisor of 4 and 6. Thus, $d = 2$ and $H = \langle 2 \rangle = \{0, 2, 4, 6, 8, 10\}$.

Item 4(a). (5pt) Given is a homomoprhism $f : \mathbb{Z} \to \mathbb{Z}_6$ such that $f(1) = 2$. Find $f(25)$, $\mathrm{Ker}(f)$ and $\mathrm{rng}(f)$. (Remark: $\mathrm{rng}(f) = f[\mathbb{Z}]$.)

Sample solution. By the homomorphism property,

$$f(n) = f(\underbrace{1 + \ldots + 1}_{n \text{ times}}) = \underbrace{f(1) +_6 \ldots +_6 f(1))}_{n \text{ times}} = \underbrace{2 +_6 \ldots +_6 2}_{n \text{ times}}.$$

It follows that $f(n) = 2n$ modulo 6. Thus, $f(25) = 2.25$ modulo 6, which is 50 modulo 6, which is 2. Thus, $f(25) = 2$.

By the theorem from the lecture, $\mathrm{Ker}(f)$ is a subgroup of $\mathbb{Z}$. Since $\mathbb{Z}$ is infinite and $\mathbb{Z}_6$ is finite, f is not injective, so $\mathrm{Ker}(f)$ is a nontrivial subgroup of $\mathbb{Z}$. And, because $\mathbb{Z}$ is cyclic, so is $\mathrm{Ker}(f)$. By the theorem from the lecture, $\mathrm{Ker}(f) = \langle a \rangle$ where a is the least positive element of $\mathrm{Ker}(f)$, i.e. the least positive number that is sent by f to 0. Now $f(1) = 2$, $f(2) = 4$ and $f(3) = 0$, so $a = 3$. Thus, $\mathrm{Ker}(f) = \langle 3 \rangle$.

By the theorem from the lecture, $\mathrm{rng}(f)$ is a subgroup of $\mathbb{Z}_3$. Again, this group is cyclic, and $f(1) = 2$ is a generator of $\mathrm{rng}(f)$. Thus, $\mathrm{rng}(f) = \langle 2 \rangle = \{0, 2, 4\}$.

Item 4(b). (5pt) Given is a homomorphism $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ such that $f(\langle 1, 0 \rangle) = 1$ and $f(\langle 1, 1 \rangle) = 2$. Find $f(\langle 13, 5 \rangle)$ and $\mathrm{Ker}(f)$.

Sample solution. Since f is a homomorphism, it preserves the operations. We first need to know $f(\langle 1, 0 \rangle)$ and $f(\langle 0, 1 \rangle)$. The former is given, the latter is obtained as follows:

$$\begin{aligned} f(\langle 0,1\rangle) &= f(\langle 1,1\rangle - \langle 1,0\rangle) = f(\langle 1,1\rangle) - f(\langle 1,0\rangle) \\ &= 2-1=1. \end{aligned}$$

Then

$$\begin{aligned} f(\langle a,b\rangle) &= f(\langle a,0\rangle + \langle 0,b\rangle) = f(\langle a,0\rangle) + f(\langle 0,b\rangle) \\ &= a\cdot f(\langle 1,0\rangle) + b\cdot f(\langle 0,1\rangle) = a+b. \end{aligned}$$

Thus, $f(\langle 13,5\rangle) = 13+5 = 18$.

Now $\langle a,b\rangle \in \mathrm{Ker}(f)$ if and only if $f(\langle a,b\rangle)=0$ and this holds if and only if $a+b=0$. The last statement is true if and only if $a=-b$. Thus,

$$\mathrm{Ker}(f) = \{\langle a,b\rangle \in \mathbb{Z}\times\mathbb{Z} \mid a=-b\}.$$

Item 5(a). (5pt) Given is the following permutation $\pi \in S_8$:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 1 & 8 & 3 & 2 & 6 & 7 \end{pmatrix}.$$

Express π both as a product of disjoint cycles and a product of permutations. Determine $\mathrm{sgn}(\pi)$.

Sample solution. The permutation π acts as follows: $1\mapsto 5\mapsto 3\mapsto 1$ and $2\mapsto 4\mapsto 8\mapsto 7\mapsto 6\mapsto 2$, so $\pi = (1,5,3)(2,4,8,7,6)$.

By the theorem from the lecture, the two above cycles can be expressed as:

$$\begin{aligned} (1,5,3) &= (3,1)(5,1) \\ (2,4,8,7,6) &= (6,2)(7,2)(8,2)(4,2) \end{aligned}$$

Thus, $\pi = (3,1)(5,1)(6,2)(7,2)(8,2)(4,2)$.

Item 5(b). (8pt) Given are the following permutations $\sigma, \sigma' \in S_5$.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix} \quad \sigma' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 1 & 4 \end{pmatrix}.$$

Compute $\langle\sigma\rangle$, σ^{50} and $\sigma\sigma'$.

Sample solution. $\langle\sigma\rangle$ is a finite cyclic group, because S_5 is finite. Thus, $\langle\sigma\rangle = \{\sigma, \sigma^2, \ldots, \sigma^n\}$ where n is the least number ≥ 0 such that $\sigma^n = \text{id}$. We now compute:

$$\begin{aligned}
\sigma &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix} \\
\sigma^2 &= \sigma\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 4 & 2 \end{pmatrix} \\
\sigma^3 &= \sigma^2\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 3 & 1 & 5 \end{pmatrix} \\
\sigma^4 &= \sigma^3\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 2 & 4 & 3 \end{pmatrix} \\
\sigma^5 &= \sigma^4\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix} \\
\sigma^6 &= \sigma^5\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}
\end{aligned}$$

Since $\sigma^6 = \text{id}$ and $\sigma^i \neq \text{id}$ for all $i \in \{1, 2, 3, 4, 5\}$, we have

$$\langle\sigma\rangle = \{\text{id}, \sigma, \sigma^2, \sigma^3, \sigma^4, \sigma^5\}.$$

Since the order of $\langle\sigma\rangle$ is 6, we have: $\sigma^{50} = \sigma^{6.8+2} = (\sigma^6)^8\sigma^2 = \sigma^2$. Thus,

$$\sigma^{50} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 4 & 2 \end{pmatrix}$$

Finally,

$$\sigma\sigma' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix}$$

Item 5(c). (7pt) Given is a homomorphism $f : \mathbb{Z}_4 \to S_4$ such that $f(1) = (1,3)(2,4)$. Find $f(3)$ and $\text{Ker}(f)$.

Sample solution. Because f is a homomorphism, $f(n) = f(1)^n$ for every $n \in \mathbb{Z}_4$. Since the cycles $(1,3)$ and $(2,4)$ are disjoint, they commute. It follows that

$$f(n) = [(1,3)(2,4)]^n = (1,3)^n(2,4)^n.$$

Now $(1,3)^3 = (1,3)^{2+1} = (1,3)^2(1,3) = \mathrm{id}(1,3) = (1,3)$. Similarly, $(2,4)^3 = (2,4)^{2+1} = (2,4)^2(2,4) = \mathrm{id}(2,4) = (2,4)$. Thus, $f(3) = (1,3)^3(2,4)^3 = (1,3)(2,4)$.

We know that $\mathrm{Ker}(f)$ is a subgroup of $\mathbb{Z}_4$, so again, $\mathrm{Ker}(f)$ is cyclic since $\mathbb{Z}_4$ is cyclic. But $\mathrm{Ker}(f)$ is nontrivial, since $f(2) = (1,3)^2(2,4)^2 = \mathrm{id}$, so $2 \in \mathrm{Ker}(f)$. Thus, $\mathrm{Ker}(f)$ is of the form $\langle a \rangle$ where a is the least positive element of $\mathrm{Ker}(f)$. But $1 \notin \mathrm{Ker}(f)$, as $f(1) = (1,3)(2,4) \neq \mathrm{id}$, and we have just seen that $2 \in \mathrm{Ker}(f)$. So $\mathrm{Ker}(f) = \langle 2 \rangle = \{0, 2\}$.