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CENTRAL CONFIGURATIONS AND TOTAL COLLISIONS FOR QUASIHOMOGENEOUS N-BODY PROBLEMS

ABSTRACT. We consider n -body problems given by potentials of the form $\frac{\alpha}{r^a} + \frac{\beta}{r^b}$ with a, b, α, β constants, $0 \leq a < b$. To analyze the dynamics of the problem, we first prove some properties related to central configurations, including a generalization of Moulton's theorem. Then we obtain several qualitative properties for collision and near-collision orbits in the Manev-type case $a = 1$. At the end we point out some new relationships between central configurations, relative equilibria, and homothetic solutions.

1. INTRODUCTION

The n -body problem studied here is given by a potential of the form $\frac{\alpha}{r^a} + \frac{\beta}{r^b}$, where r is the distance between bodies and a, b, α, β are constants, $0 \leq a < b$ (see [4, 12]). In the first part of the paper we treat the general problem, and in the second part we focus on the case $a = 1$. The function $\frac{\alpha}{r^a} + \frac{\beta}{r^b}$, called quasihomogeneous because of being the sum of homogenous functions of different degrees, generalizes classical potentials, such as those of Newton, Coulomb, Birkhoff, Manev, Van der Waals, Libhoff, Schwarzschild, and Lennard-Jones. Thus, the applicability of the quasihomogeneous n -body problem ranges from celestial mechanics and atomic physics to chemistry and crystallography.

Although many properties of the Newtonian n -body problem have a correspondent in the homogeneous case, this is not true for nonhomogeneous potentials. On one hand, the transposition of known results is far from trivial; on the other, new properties show up.

An intriguing aspect we will point out in this paper refers to central configurations, which are crucial for understanding the dynamics of the n -body problem (see [13]). The central configurations of the quasihomogeneous potential are in a certain relationship with the central configurations of the homogeneous functions that form this potential. Thus, we will introduce here the notion of simultaneous central configuration and will investigate its connection with the classical concept.

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In Section 2, we define the quasihomogeneous n -body problem and write down the equations of motion. In Section 3, we introduce the concepts of central and simultaneous central configuration, the latter being specific to quasihomogeneous potentials. Section 4 deals with collinear central configurations. Using critical point theory, we prove a generalization of Moulton's theorem by showing that the number of collinear central configurations of n bodies is $n!/2$. Starting with Section 5, we restrict our study to Manev-type problems, [5], i.e. those given by potentials of the form $\frac{\alpha}{r} + \frac{\beta}{r^b}$, and show that there are exactly two planar central configurations in the 3-body case. Section 6 introduces a framework for the study of collision and near-collision orbits, which is performed in Sections 7 and 8. We study in detail the network of collision solutions and determine the relationship between central configurations, on one hand, and relative equilibria and homothetic orbits, on the other hand. A remarkable fact is that, for Manev-type potentials, homothetic orbits are less likely than in the Newtonian case, in the sense that they show up only for simultaneous central configurations.

2. THE QUASIHOMOGENEOUS n -BODY PROBLEM

We will start with defining the planar quasihomogeneous n -body problem. Consider the linear space

$$(1) \quad \Omega = \{\mathbf{r} = (\mathbf{r}_1, \dots, \mathbf{r}_n) \in (\mathbb{R}^2)^n \mid \sum_{i=1}^n m_i \mathbf{r}_i = 0\},$$

where $m_i > 0, i = 1, 2, \dots, n$, are the masses of the n bodies and $\mathbf{r}_i, i = 1, 2, \dots, n$, represent their coordinates. Notice that $\sum_{i=1}^n m_i \mathbf{r}_i = 0$ fixes the centre of mass at the origin of the coordinate system. Let

$$(2) \quad \Delta_{ij} = \{(\mathbf{r}_1, \dots, \mathbf{r}_n) \in \Omega \mid \mathbf{r}_i = \mathbf{r}_j\}; \quad \Delta = \bigcup_{i,j} \Delta_{ij}.$$

We call Δ the collision set. The potential U of the system is a function defined on the configuration space $\tilde{\Omega} = \Omega \setminus \Delta$ and is given by

$$U = W + V,$$

where W is a homogeneous function of degree $-a, a \geq 0$,

$$(3) \quad W(\mathbf{r}_1, \dots, \mathbf{r}_n) = \sum_{i < j} \frac{m_i m_j}{\|\mathbf{r}_i - \mathbf{r}_j\|^a},$$

and V is a homogeneous function of degree $-b, b > a$,

$$(4) \quad V(\mathbf{r}_1, \dots, \mathbf{r}_n) = \sum_{i < j} \frac{m_i m_j}{\|\mathbf{r}_i - \mathbf{r}_j\|^b}.$$

The equations of motion of the n bodies define a vector field X on the tangent bundle $T(\tilde{\Omega})$. The configuration space of the system is $\tilde{\Omega}$ and the cotangent bundle is $T^*(\tilde{\Omega})$. Let $\mathbf{p} = M^{-1}\dot{\mathbf{r}}$ be the linear momentum of the system of particles, where M is the diagonal matrix $M = \text{diag}(m_1, m_1, m_2, m_2, \dots, m_n, m_n)$. Then the equations of motion can be written as a Hamiltonian system,

$$(5) \quad \begin{aligned} \dot{\mathbf{r}} &= \frac{\partial H}{\partial \mathbf{p}} \\ \dot{\mathbf{p}} &= -\frac{\partial H}{\partial \mathbf{r}}, \end{aligned}$$

where $H : T^*(\tilde{\Omega}) \rightarrow \mathbb{R}$ is the Hamiltonian function given by

$$(6) \quad H(\mathbf{r}, \mathbf{p}) = \frac{1}{2} \mathbf{p}^t M^{-1} \mathbf{p} - U(\mathbf{r}).$$

Here $T = \frac{1}{2} \mathbf{p}^t M^{-1} \mathbf{p}$ is the kinetic energy. The total energy H is a first integral for the system (5); this means that $T - U = h$ (constant) along any orbit. Other integrals are given by the linear momentum, $\sum_{i=1}^n m_i \dot{\mathbf{r}}_i$, and by the angular momentum, $J : T \rightarrow \mathbf{R}$, defined as

$$(7) \quad J(\mathbf{r}, \mathbf{v}) = \sum_{i=1}^n m_i \mathbf{r}_i \times \mathbf{v}_i.$$

Notice that the relationships for the centre of mass, $\sum_{i=1}^n m_i \mathbf{r}_i = 0$, and linear momentum, $\sum_{i=1}^n m_i \dot{\mathbf{r}}_i = 0$, together with the energy integral, $T - U = h$, reduce the dimension of the Hamiltonian system (5) from $4n$ to $4n - 5$. We also introduce the scalar product,

$$(8) \quad \langle \mathbf{r}, \tilde{\mathbf{r}} \rangle = \mathbf{r}^t M \tilde{\mathbf{r}},$$

which allows us to write the moment of inertia as

$$(9) \quad I = \langle \mathbf{r}, \mathbf{r} \rangle = \sum_{i=1}^n m_i \|\mathbf{r}_i\|^2.$$

3. CENTRAL CONFIGURATIONS

Central configurations play a crucial role for understanding the dynamics of n -body problems [13]. In particular, they have led to important theoretical investigations, such as Saari's conjecture, which has remained open for more than three and a half decades [6], and are connected to Smale's 6th problem [15], originally proposed by Wintner in 1941, [16] (see also [9, 11]). In this section we will define central configurations and analyze the particular aspects this concept encounters in the quasihomogeneous case.

Definition 1. A configuration $\mathbf{r} \in \tilde{\Omega}$ is called central if there is a constant σ such that

$$(10) \quad \nabla U(\mathbf{r}) = \sigma \nabla I(\mathbf{r}).$$

Using the fact that the functions W and V are homogeneous of degree $-a$ and $-b$, respectively, and applying Euler's theorem for homogeneous functions, we find that

$$(11) \quad \sigma = \frac{-aW(\mathbf{r}) - bV(\mathbf{r})}{2I(\mathbf{r})}.$$

Definition 2. We call $\mathbf{r} \in \tilde{\Omega}$ a simultaneous central configuration for the potentials W and V if there are constants σ_1 and σ_2 such that

$$\nabla W(\mathbf{r}) = \sigma_1 \nabla I(\mathbf{r}) \quad \text{and} \quad \nabla V(\mathbf{r}) = \sigma_2 \nabla I(\mathbf{r}).$$

Using the fact that W and V are homogeneous functions of degree $-a$ and $-b$, respectively, we find that

$$(12) \quad \sigma_1 = \frac{-aW(\mathbf{r})}{2I(\mathbf{r})} \quad \text{and} \quad \sigma_2 = \frac{-bV(\mathbf{r})}{2I(\mathbf{r})}.$$

Note that if \mathbf{r} is a simultaneous central configuration for W and V , then \mathbf{r} is also a central configuration for $U = V + W$. The converse is not necessarily true.

Let

$$S_{I_0} = \{\mathbf{r} \in \Omega \mid \langle \mathbf{r}, \mathbf{r} \rangle = I_0\}$$

be the sphere relative to the metric given by the scalar product, and denote by

$$S_{I_0}^* = S_{I_0} \setminus \Delta = \{\mathbf{r} \in \tilde{\Omega} \mid \langle \mathbf{r}, \mathbf{r} \rangle = I_0\}$$

this sphere minus the collision set. Then the central configurations with moment of inertia I_0 can also be defined as the critical points of U_{S_I} , where $U_{S_I} : S_I^* \rightarrow \mathbb{R}$ is the restriction of the potential U to $S_{I_0}^*$. Denote by C_n the set of central configurations of the quasihomogeneous n -body problem.

Definition 3. We say that two relative equilibria in $S_{I_0}^*$ are equivalent (and belong to the same equivalence class) if they can be made congruent by the induced S^1 action on $S_{I_0}^*$, that is, if one is obtained from the other by a rotation.

Let \tilde{C}_n denote the set of equivalence classes of central configurations. Note that this definition differs from the one used in the Newtonian case (see [1, 14]), where two central configuration are called equivalent when one can be obtained from the other by a rotation and/or a homothety. This change is necessary in the quasihomogeneous case because the set

C_n is invariant under the action of the group S^1 , but not necessarily under the action of homotheties (see Section 7).

Clearly, I and Δ are invariant under the action of S^1 . Thus, we can conclude that $S_{I_0}^*$ is diffeomorphic to the $(2n - 3)$ -dimensional sphere S^{2n-3} (which is actually an ellipsoid E^{2n-3}) with all the points Δ removed, that is,

$$S_{I_0}^* = E^{2n-3} \setminus (E^{2n-3} \cap \Delta) \approx S^{2n-3} \setminus (S^{2n-3} \cap \Delta).$$

Since U_{S_I} is invariant under the action of S^1 , it defines a map $\tilde{U}_{S_I} : S_{I_0}^*/S^1 \rightarrow \mathbb{R}$. If we let $\pi_n : S_{I_0}^* \rightarrow S_{I_0}^*/S^1$ denote the canonical projection, $\tilde{\Delta} = \pi(E^{2n-3} \cap \Delta)$, and recalling that $E^{2n-3}/S^1 \approx S^{2n-3}/S^1 \approx \mathbb{C}P^{n-2}$ (the complex projective space), we are led to investigate the critical points of $\tilde{U}_{S_I} : \mathbb{C}P^{n-2} \setminus \tilde{\Delta} \rightarrow \mathbb{R}$.

Consequently we can show that the set of equivalence classes of central configurations with fixed moment of inertia I_0 is given by the set of critical points of the map $\tilde{U}_{S_I} : \mathbb{C}P^{n-2} \setminus \tilde{\Delta} \rightarrow \mathbb{R}$. More precisely, we have proved the following property:

Proposition 1. *For any choice of masses in the planar n -body problem with a quasihomogeneous potential, $n \geq 2$, the set of equivalence classes of central configurations with moment of inertia I_0 is diffeomorphic with the set of critical points of the map $\tilde{U}_{S_I} : \mathbb{C}P^{n-2} \setminus \tilde{\Delta} \rightarrow \mathbb{R}$.*

4. MOULTON'S THEOREM FOR QUASIHOMOGENEOUS POTENTIALS

We will now study collinear central configurations and, using critical point theory, will calculate the number of classes of such configurations for any number n of bodies. The goal of this section is to prove the following result, which generalizes a theorem obtained by Forest Ray Moulton in 1910, [10].

Theorem 1. *For any choice of masses in the n -body problem with a quasihomogeneous potential, U , and any given moment of inertia, I_0 , there are exactly $n!/2$ classes of collinear central configurations. In other words, there are $n!/2$ classes of central configurations $\mathbf{r} = (\mathbf{r}_1, \dots, \mathbf{r}_n)$, where all \mathbf{r}_i belong to the same straight line through the origin.*

In preparation for the proof, choose some line l in \mathbb{R}^2 . This defines a subset $\Omega_l \subset \Omega$ of $\mathbf{r} = (\mathbf{r}_1, \dots, \mathbf{r}_n)$ such that each \mathbf{r}_i is on the line l . Let $S_l = S_{I_0} \cap \Omega_l$ and $S_l^* = S_l \setminus \cap(S_l \cap \Delta)$. When S^1 acts on S_{I_0} , only the rotation by π radians leaves S_l invariant. Thus the group \mathbb{Z}_2 acts on S_l , and on the quotient we have $\mathbb{R}P^{n-2} \setminus \tilde{\Delta} \subset \mathbb{C}P^{n-2} \setminus \tilde{\Delta} \xrightarrow{\tilde{U}_{S_I}} \mathbb{R}$,

where $\mathbb{R}P^{n-2} = S_l \setminus \mathbb{Z}_2$ is the real projective space, naturally contained in $\mathbb{C}P^{n-2}$. Here \tilde{U}_{S_l} is induced by the potential energy. From these considerations we obtain:

Lemma 1. *The set of equivalence classes of collinear central configurations with moment of inertia I_0 is diffeomorphic to the set of critical points of $\tilde{U}_{S_l} : \mathbb{C}P^{n-2} \setminus \tilde{\Delta} \rightarrow \mathbb{R}$ that lie in $\mathbb{R}P^{n-2} \setminus \tilde{\Delta} \subset \mathbb{C}P^{n-2} \setminus \tilde{\Delta}$.*

So in order to describe the collinear central configurations, it is sufficient to obtain the critical points of the potential that lie in the real projective space. In general, a critical point of a function restricted to a submanifold is not necessarily a critical point of the function on the ambient manifold. However, we have the following result:

Proposition 2. *If $\mathbf{r} \in \mathbb{R}P^{n-2} \setminus \tilde{\Delta}$ is a critical point of $\tilde{U}_{S_l} : \mathbb{R}P^{n-2} \setminus \tilde{\Delta} \rightarrow \mathbb{R}$, then \mathbf{r} is also a critical point of $\tilde{U}_{S_l} : \mathbb{C}P^{n-2} \setminus \tilde{\Delta} \rightarrow \mathbb{R}$.*

To prove this, we first need to know the derivatives of the potential function, which are given below.

Lemma 2. *For given masses m_1, \dots, m_n and $U = W + V$,*

(1) *The first derivative of $U : \tilde{\Omega} \rightarrow \mathbb{R}$ is*

$$\begin{aligned} DU(\mathbf{r})(\mathbf{v}) &= -a \sum_{i \neq j} \frac{m_i m_j}{\|\mathbf{r}_i - \mathbf{r}_j\|^{a+2}} (\mathbf{r}_i - \mathbf{r}_j, \mathbf{v}_i - \mathbf{v}_j) \\ &\quad - b \sum_{i \neq j} \frac{m_i m_j}{\|\mathbf{r}_i - \mathbf{r}_j\|^{b+2}} (\mathbf{r}_i - \mathbf{r}_j, \mathbf{v}_i - \mathbf{v}_j) \end{aligned}$$

for $\mathbf{v} \in \Omega$.

(2) *The 2nd derivative is*

$$\begin{aligned} D^2U(\mathbf{r})(\mathbf{v}, \mathbf{w}) &= a \sum_{i \neq j} \frac{m_i m_j}{\|\mathbf{r}_i - \mathbf{r}_j\|^{a+2}} \\ &\quad \cdot \left(\frac{a+2}{\|\mathbf{r}_i - \mathbf{r}_j\|^2} (\mathbf{r}_i - \mathbf{r}_j, \mathbf{v}_i - \mathbf{v}_j)(\mathbf{r}_i - \mathbf{r}_j, \mathbf{w}_i - \mathbf{w}_j) - (\mathbf{v}_i - \mathbf{v}_j, \mathbf{w}_i - \mathbf{w}_j) \right) \\ &\quad + b \sum_{i \neq j} \frac{m_i m_j}{\|\mathbf{r}_i - \mathbf{r}_j\|^{b+2}} \\ &\quad \cdot \left(\frac{b+2}{\|\mathbf{r}_i - \mathbf{r}_j\|^2} (\mathbf{r}_i - \mathbf{r}_j, \mathbf{v}_i - \mathbf{v}_j)(\mathbf{r}_i - \mathbf{r}_j, \mathbf{w}_i - \mathbf{w}_j) - (\mathbf{v}_i - \mathbf{v}_j, \mathbf{w}_i - \mathbf{w}_j) \right), \end{aligned}$$

where $\mathbf{v}, \mathbf{w} \in \Omega$.

(3) *The 2nd derivative of the restriction $U : S_{I_0}^* \rightarrow \mathbb{R}$ is:*

$$D^2U/(S_{I_0}^*)(\mathbf{r})(\mathbf{v}, \mathbf{w}) = D^2U(\mathbf{r})(\mathbf{v}, \mathbf{w}) + \frac{aW(\mathbf{r}) + bV(\mathbf{r})}{I_0} \langle \mathbf{v}, \mathbf{w} \rangle.$$

Here (\cdot, \cdot) denotes the usual inner product in \mathbb{R}^2 , $\|\cdot\|$ the norm in \mathbb{R}^2 , and I the moment of inertia. The same formulas are valid in \mathbb{R} , \mathbb{R}^2 and \mathbb{R}^3 .

Proof. All the equations above can be derived by differentiating in local Cartesian coordinates. \square

Now we can give a proof of Proposition 2. For $\mathbf{v}_i \in \mathbb{R}^2$, let $\mathbf{v}_i = (v_i^\parallel, v_i^\perp)$ where $v_i^\parallel \in l$ and $v_i^\perp \in l^\perp$. Then we can write $\mathbf{v} = (\mathbf{v}^\parallel, \mathbf{v}^\perp)$ with $\mathbf{v}^\parallel = (v_1^\parallel, \dots, v_n^\parallel)$, $\mathbf{v}^\perp = (v_1^\perp, \dots, v_n^\perp)$ for each $\mathbf{v} \in \Omega$. If $\mathbf{r} \in S_l \subset S_{I_0}$, $\mathbf{r} \notin \Delta$, we have $T_{\mathbf{r}}(S_{I_0}) = \{\mathbf{v} \in \Omega \mid \langle \mathbf{v}, \mathbf{r} \rangle = 0\}$ and $T_{\mathbf{r}}(S_l) = \{\mathbf{w} \in \Omega_l \mid \langle \mathbf{w}, \mathbf{r} \rangle = 0\}$ where, as usual, Ω is endowed with the mass scalar product. If $\mathbf{v} \in T_{\mathbf{r}}(S_{I_0})$ and $\mathbf{v} = (\mathbf{v}^\parallel, \mathbf{v}^\perp)$, then $\mathbf{v}^\parallel \in \Omega_l$ and $\langle \mathbf{v}, \mathbf{r} \rangle = \langle \mathbf{v}^\parallel, \mathbf{r} \rangle$. Thus $\mathbf{v}^\parallel \in T_{\mathbf{r}}(S_l)$, because $\langle \mathbf{v}, \mathbf{r} \rangle = 0$ implies $\langle \mathbf{v}^\parallel, \mathbf{r} \rangle = 0$.

By Lemma 2 it follows that if $\mathbf{r} \in S_l \setminus \Delta$ and $\mathbf{v} \in T_{\mathbf{r}}(S_{I_0})$, then $DU(\mathbf{r})(\mathbf{v}) = DU(\mathbf{r})(\mathbf{v}^\parallel)$. So $DU(\mathbf{r})(\mathbf{v}^\parallel) = 0$ implies that $DU(\mathbf{r})(\mathbf{v}) = 0$. This completes the proof of Proposition 2.

Lemma 3. $\mathbb{R}P^{n-2}$ has $n!/2$ components.

Proof. Let $\mathbf{r} = (\mathbf{r}_1, \dots, \mathbf{r}_2) \in S_l \setminus \Delta$ and let $\mathbf{r}_1 < \dots < \mathbf{r}_n \in \mathbb{R}$ (we use the fact that the \mathbf{r}_i are all distinct). Let $\alpha = (i_1, \dots, i_n)$ be an arbitrary permutation of the numbers $(1, 2, \dots, n)$. If we apply the permutation to the initial vector \mathbf{r} , we map it to a different component defined uniquely by the given permutation. Therefore the set $S_l \setminus \Delta$ has $n!$ components and the quotient space $\mathbb{R}P^{n-2} \setminus \Delta$ has $n!/2$ components. \square

We can now prove Moulton's theorem for quasihomogeneous potentials. By applying part (2) and (3) of Lemma 2, we see that $D^2U/(S_l \setminus \Delta)$ is a positive definite form, and consequently \tilde{U} is convex. This shows that \tilde{U} has a unique minimum in each component of $\mathbb{R}P^{n-2}$. Thus there are $n!/2$ critical points and hence $n!/2$ central configurations.

Remark 1. We have identified the symmetric central configurations, otherwise the number of classes of central configurations would be $n!$.

5. PLANAR CENTRAL CONFIGURATIONS

In this and subsequent sections, we will restrict our study to Manev-type quasihomogeneous potentials, namely those U for which $a = 1$ (see also [5]). They form an important class of quasihomogeneous potentials, derived from the Manev law, which can explain the perihelion

advance of the planet Mercury within the framework of classical mechanics (for more details see [4] and [3]). Since for any planar central configuration in the Manev-type three body problem, the mutual distances are geometrically independent, we can solve the equations defining the central configurations in terms of the mutual distances. To be precise, we state here a result whose proof can be found in [2].

Lemma 4. *Let $u = f(\mathbf{x})$ be a function with $\mathbf{x} = (x_1, x_2, \dots, x_n)$, $x_1 = g_1(\mathbf{y})$, $x_2 = g_2(\mathbf{y})$, \dots , $x_n = g_n(\mathbf{y})$, $\mathbf{y} = (y_1, y_2, \dots, y_m)$ and $m \geq n$.*

If $\text{rank}(A) = n$, where

$$(13) \quad A = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_1}{\partial y_m} & \cdots & \frac{\partial x_n}{\partial y_m} \end{pmatrix},$$

then $\nabla f(\mathbf{x}) = \mathbf{0}$ if and only if $\nabla u(\mathbf{y}) = \mathbf{0}$.

Let us now consider the 3-body case, and for this purpose we will use the notation $\mathbf{r}_i = (q_{i1}, q_{i2})$ for $i = 1, 2, 3$. From Lemma 4 we have that if $\text{rank}(A) = 3$, where

$$A = \begin{pmatrix} \frac{\partial r_{12}}{\partial q_{11}} & \frac{\partial r_{13}}{\partial q_{11}} & \frac{\partial r_{23}}{\partial q_{11}} \\ \frac{\partial r_{12}}{\partial q_{12}} & \frac{\partial r_{13}}{\partial q_{12}} & \frac{\partial r_{23}}{\partial q_{12}} \\ \frac{\partial r_{12}}{\partial q_{21}} & \frac{\partial r_{13}}{\partial q_{21}} & \frac{\partial r_{23}}{\partial q_{21}} \\ \frac{\partial r_{12}}{\partial q_{22}} & \frac{\partial r_{13}}{\partial q_{22}} & \frac{\partial r_{23}}{\partial q_{22}} \\ \frac{\partial r_{12}}{\partial q_{31}} & \frac{\partial r_{13}}{\partial q_{31}} & \frac{\partial r_{23}}{\partial q_{31}} \\ \frac{\partial r_{12}}{\partial q_{32}} & \frac{\partial r_{13}}{\partial q_{32}} & \frac{\partial r_{23}}{\partial q_{32}} \end{pmatrix} = \begin{pmatrix} \frac{q_{11}-q_{21}}{r_{12}} & \frac{q_{11}-q_{31}}{r_{13}} & 0 \\ \frac{q_{12}-q_{22}}{r_{12}} & \frac{q_{12}-q_{32}}{r_{13}} & 0 \\ -\frac{q_{11}-q_{21}}{r_{12}} & 0 & \frac{q_{21}-q_{31}}{r_{23}} \\ -\frac{q_{12}-q_{22}}{r_{12}} & 0 & \frac{q_{22}-q_{32}}{r_{23}} \\ 0 & -\frac{q_{11}-q_{31}}{r_{13}} & -\frac{q_{21}-q_{31}}{r_{23}} \\ 0 & -\frac{q_{12}-q_{32}}{r_{13}} & -\frac{q_{22}-q_{32}}{r_{23}} \end{pmatrix},$$

then $\nabla U(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \mathbf{0}$ if and only if $\nabla U(r_{12}, r_{13}, r_{23}) = 0$.

Some straightforward computations show that the $\text{rank}(A) = 3$ if and only if

$$\det \begin{pmatrix} q_{11} & q_{12} & 1 \\ q_{21} & q_{22} & 1 \\ q_{31} & q_{32} & 1 \end{pmatrix} \neq 0.$$

This determinant is twice the oriented area of the triangle formed by the 3 particles. In short, if \mathbf{r}_1 , \mathbf{r}_2 and \mathbf{r}_3 are not collinear, then $\nabla U(r_{12}, r_{13}, r_{23}) = \mathbf{0}$ if and only if $\nabla U(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \mathbf{0}$.

Using Lemma 4 in order to find the planar central configurations, we first need to solve the equation

$$(14) \quad \nabla U = \sigma \nabla I$$

in terms of the mutual distances r_{ij} , taking into account the fact that the moment of inertia, I , can be written in terms of the mutual distances as $I = (1/\tilde{m}) \sum_{i=1}^n m_i m_j r_{ij}^2$, where \tilde{m} is the total mass. So, for fixed i and j , we have

$$-\frac{m_i m_j}{r_{ij}^2} - b \frac{m_i m_j}{r_{ij}^{b+1}} = 2 \frac{\sigma}{\tilde{m}} m_i m_j r_{ij}.$$

Multiplying by r_{ij}^{b+1} , we obtain

$$f(r_{ij}) := 2\sigma r_{ij}^{b+2} + \tilde{m} r_{ij}^{b-1} + \tilde{m} b = 0.$$

Regarding the above equation as a polynomial in the variable r_{ij} , since $\sigma < 0$, $f(0) = b\tilde{m} > 0$ and the coefficients polynomial have just one change of sign, we can verify easily that the function f has exactly one positive root. Observe that the function f only depends on the total mass \tilde{m} , and therefore the respective solution for $f(r_{ij})$ is the same for all mutual distances. We have thus proved the following result.

Theorem 2. *In the Manev-type three body problem, for any values of the masses, there are exactly two equilateral central configurations, which correspond to the two possible orientations of a triangle in a plane.*

6. A FRAMEWORK FOR THE STUDY OF COLLISIONS

We will further study the dynamics at an near total collision for Manev-type n -body problems. A convenient framework for this purpose is given by the so-called McGehee coordinates [8],

$$\begin{aligned} \rho &= (\mathbf{r}^t M \mathbf{r})^{1/2} \\ \mathbf{s} &= \rho^{-1} \mathbf{r} \\ v &= \rho^{b/2} (\mathbf{p}^t \mathbf{s}) \\ \mathbf{u} &= \rho^{b/2} (\mathbf{p} - (\mathbf{p}^t \mathbf{s}) M \mathbf{s}), \end{aligned} \tag{15}$$

where $M = \text{diag}(m_1, m_1, m_2, m_2, \dots, m_n, m_n)$, $\mathbf{r} = (\mathbf{r}_1, \dots, \mathbf{r}_n)$, and $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_n)$. After a reparametrization of the time variable,

$$d\tau = r^{-1-b/2} dt, \tag{16}$$

the equations of motion (5) become

$$\begin{aligned}
 \rho' &= \rho v \\
 v' &= \frac{b}{2}v^2 + \mathbf{u}^t M^{-1} \mathbf{u} - \rho^{b-1} W(\mathbf{s}) - bV(\mathbf{s}) \\
 \mathbf{s}' &= M^{-1} \mathbf{u} \\
 \mathbf{u}' &= \left(\frac{b}{2} - 1 \right) \mathbf{u} v - (\mathbf{u}^t M^{-1} \mathbf{u}) M \mathbf{s} + \\
 &\quad \rho^{b-1} [W(\mathbf{s}) M \mathbf{s} + \nabla W(\mathbf{s})] + bV(\mathbf{s}) M \mathbf{s} + \nabla V(\mathbf{s}).
 \end{aligned}
 \tag{17}$$

Here the prime denotes differentiation with respect to the new (fictitious) time variable τ , and the old notation is maintained for the new dependent variables, which are now functions of τ . Furthermore, the new variables fulfill the constraints $\mathbf{s}^t M \mathbf{s} = 1$ and $\mathbf{u}^t \mathbf{s} = 0$.

In these coordinates the energy integral (6) turns into the relation

$$\frac{1}{2}(\mathbf{u}^t M^{-1} \mathbf{u} + v^2) - \rho^{b-1} W(\mathbf{s}) - V(\mathbf{s}) = h\rho^b.
 \tag{18}$$

We define the total collision manifold as

$$C = \{(\rho, \mathbf{s}, v, \mathbf{u}) \mid \rho = 0, \quad \mathbf{u}^t M^{-1} \mathbf{u} + v^2 - 2V(\mathbf{s}) = 0\}.
 \tag{19}$$

Notice that $\rho' = 0$ if $\rho = 0$, so C (which is an analytic submanifold of codimension 1 in the boundary of the phase space) is invariant under the flow of the system (17). By continuity of the solutions respect to initial conditions, the flow on C provides important information about the orbits close to triple collision (see [8] for more details). The total collision manifold can also be regarded as an invariant boundary pasted onto each energy surface:

$$E_h = \{(\rho, v, \mathbf{s}, \mathbf{u}) \mid \frac{1}{2}(\mathbf{u}^t M^{-1} \mathbf{u} + v^2) - \rho^{b-1} W(\mathbf{s}) - V(\mathbf{s}) = h\rho^b\}.
 \tag{20}$$

These concepts are ideal for understanding the qualitative behaviour of total- and near-total-collision solutions.

7. COLLISION AND NEAR-COLLISION DYNAMICS

In this section we will study the dynamics of total- and near-total-collision orbits of the Manev-type n -body problem. An important role in this study is played by central configurations and by the solutions that can be derived from them.

In the planar Newtonian n -body problem, a rigid rotation of a central configuration is called a relative equilibrium; in rotating coordinates, relative equilibria are fixed points. A non-rotating homothetic orbit of

a central configuration is called a homothety. The composition of a relative equilibrium and a homothety is called a homographic solution.

In the Manev-type three-body problem, since the potential only depends of the bodies' mutual distances, the central configurations are invariant under rotations, so any central configuration determines a particular periodic orbit, which in a rotating frame is a fixed point. So in the Manev-type three body problem, any central configuration corresponds to a relative equilibrium.

In the Newtonian case, any central configuration also corresponds to a homothetic orbit. But is this valid for Manev-type potentials too? As we will further prove (see Theorem 5 and Section 8), this property is not satisfied in general. To show this, and to determine under what circumstances homothetic solutions still exist, we will prove several preliminary results.

Notice that the flow on C is given by the equations

$$(21) \quad \begin{aligned} v' &= \frac{b}{2}v^2 + \mathbf{u}^t M^{-1} \mathbf{u} - bV(\mathbf{s}) \\ \mathbf{s}' &= M^{-1} \mathbf{u} \\ \mathbf{u}' &= \left(\frac{b}{2} - 1\right) \mathbf{u} v - (\mathbf{u}^t M^{-1} \mathbf{u}) M \mathbf{s} + bV(\mathbf{s}) M \mathbf{s} + \nabla V(\mathbf{s}). \end{aligned}$$

The equilibrium points of system (21) are given by $\mathbf{u} = 0$, $v = \pm \sqrt{2V(\mathbf{s})}$, where \mathbf{s} must be a critical point for the function $V(\mathbf{s})$ restricted to the unit sphere corresponding to the mass matrix M . The masses are involved because the equation $bV(\mathbf{s}) M \mathbf{s} + \nabla V(\mathbf{s}) = 0$ must be satisfied. But these are the critical points of the function \tilde{V} , which is the restriction of the homogeneous potential V to the unit sphere given by the mass matrix M . Such critical points correspond to the central configurations of the homogeneous potential V .

Proposition 3. *For any value of $b > 2$, the flow on the total collision manifold C is gradient-like with respect to the coordinate $-v$ (i.e. the flow increases with respect to $-v$ along non-equilibrium solutions).*

Proof. The energy relation (18), restricted to C , takes the form

$$\mathbf{u}^t M^{-1} \mathbf{u} + v^2 - 2V(\mathbf{s}) = 0.$$

Using the above expression and (21), we get that

$$(22) \quad v' = \left(1 - \frac{b}{2}\right) \mathbf{u}^t M^{-1} \mathbf{u}$$

on C . If $\mathbf{u} \neq \mathbf{0}$ then $v' < 0$ is increasing with respect to $-v$. On the other hand if $\mathbf{u} = \mathbf{0}$ then $\mathbf{u}' = bV(\mathbf{s}) M \mathbf{s} + \nabla V(\mathbf{s})$, i.e. $\mathbf{u}' = 0$

only if \mathbf{s} is a critical point of \tilde{V} . Consequently $-v$ is strictly increasing along nonequilibrium solutions, which means that the vector field is gradient-like with respect to $-v$. \square

Denote by $\text{ind}(\mathbf{s}_0)$ the index of the critical point \mathbf{s}_0 , i.e. the number of eigenvalues of $D^2\tilde{V}(\mathbf{s}_0)$ with negative real part. Then we can prove the following result.

Theorem 3. *Let \mathbf{s}_0 be a nondegenerate central configuration of the planar n -body problem with potential V and $b > 2$. Then the dimensions of $W^u(\mathbf{s}_0^+)$ and $W^s(\mathbf{s}_0^-)$ are the same and equal to $2n - 2 - \text{ind}(\mathbf{s}_0)$ in E_h . The dimensions of $W^s(\mathbf{s}_0^+)$ and $W^u(\mathbf{s}_0^-)$ are the same and equal to $2n - 4 + \text{ind}(\mathbf{s}_0)$ in E_h . The dimension of E_h is $4n - 5$.*

Proof. Let \mathbf{s}_0 be a central configuration, $v = \pm\sqrt{2V(\mathbf{s}_0)}$, and $\mathbf{u} = 0$, then the equation of motion restricted to E_h are

$$\begin{aligned}
\rho' &= \rho v \\
v' &= \left(1 - \frac{b}{2}\right) \mathbf{u}^t M^{-1} \mathbf{u} + (b-1)\rho^{b-1}W(\mathbf{s}) + bh\rho^b \\
(23) \quad \mathbf{s}' &= M^{-1} \mathbf{u} \\
\mathbf{u}' &= \left(\frac{b}{2} - 1\right) \mathbf{u}v - (\mathbf{u}^t M^{-1} \mathbf{u})M\mathbf{s} + \\
&\quad \rho^{b-1}[W(\mathbf{s})M\mathbf{s} + \nabla W(\mathbf{s})] + bV(\mathbf{s})M\mathbf{s} + \nabla V(\mathbf{s}).
\end{aligned}$$

Taking into account the centre of mass and linear momentum integrals as well as the restrictions $\mathbf{s}^t M \mathbf{s} = 1$ and $\mathbf{u}^t \mathbf{s} = 0$ of the McGehee coordinates, the above system has dimension $4n - 4$.

Linearizing the system, the eigenvalues for $b > 2$ are given by the matrix equation

$$(24) \quad \begin{pmatrix} v & 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & * & \dots & \dots & * \\ \vdots & \vdots & & O_{2n-3} & I_{2n-3} & \\ \vdots & \vdots & & A & (\frac{b}{2} - 1)vI_{2n-3} & \\ 0 & 0 & & & & \end{pmatrix} - \mu I_{4n-4} = O_{4n-4},$$

where I_N is the $N \times N$ identity matrix, O_N is the $N \times N$ zero matrix, A denotes the Hessian matrix of \tilde{V} (i.e. the potential restricted to the sphere of constant moment of inertia) and $*$ denotes an element without importance in the computation of the eigenvalues.

It is clear that the first two eigenvalues are $v \neq 0$ (since $V(\mathbf{s}_0) \neq 0$) and 0. To obtain the remaining eigenvalues of equation (24), suppose

\mathbf{z} is a $(2n - 3)$ -vector satisfying

$$(25) \quad A\mathbf{z} = \lambda_i \mathbf{z}$$

for $i = 1, \dots, 2n - 3$, i fixed, where $\lambda_1, \dots, \lambda_{2n-3}$ are the eigenvalues of A . Then

$$\begin{pmatrix} O_{2n-3} & I_{2n-3} \\ A & (b/2 - 1)vI_{2n-3} \end{pmatrix} \begin{pmatrix} \mathbf{z} \\ \mu \mathbf{z} \end{pmatrix} = \begin{pmatrix} \mu \mathbf{z} \\ \{\lambda_i + (b/2 - 1)v\mu\} \mathbf{z} \end{pmatrix}.$$

Consequently μ is a root of equation (24) if

$$\mu^2 - (b/2 - 1)v\mu - \lambda_i = 0,$$

which gives

$$\mu_i^{1,2} = \frac{1}{4} \{(b - 2)v \pm \sqrt{(2 - b)^2 v^2 + 16\lambda_i}\}$$

for $i = 1, \dots, 2n - 3$.

Then if $v = \sqrt{V(\mathbf{s}_0)}$, the differential matrix of the vector field restricted to E_h has $2n - 2 - \text{ind}(\mathbf{s}_0)$ eigenvalues with positive real part and $2n - 4 + \text{ind}(\mathbf{s}_0)$ with negative real part. The values of the dimensions are switched if $v = -\sqrt{V(\mathbf{s}_0)}$. \square

Theorem 4. *Let \mathbf{s}_0 be a central configuration of the collinear n -body problem with potential V and $b > 2$. Then the dimensions of $W^u(\mathbf{s}_0^+)$ and $W^s(\mathbf{s}_0^-)$ are the same and equal to $n - 1$ in E_h . The dimensions of $W^s(\mathbf{s}_0^+)$ and $W^u(\mathbf{s}_0^-)$ are the same and equal to $n - 2$ in E_h . The dimension of E_h is $2n - 3$.*

Proof. The proof is similar to the one of the previous theorem. Let \mathbf{s}_0 be a central configuration, $v = \pm\sqrt{2V(\mathbf{s}_0)}$. The equations of motion restricted to E_h are given by equation (23), with the obvious modifications.

Linearizing the system, the eigenvalues in the case $b > 2$ are given by the following matrix equation

$$(26) \quad \begin{pmatrix} v & 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & * & \dots & \dots & * \\ \vdots & \vdots & & O_{n-2} & I_{n-2} & \\ \vdots & \vdots & & A & (\frac{b}{2} - 1)vI_{n-2} & \\ 0 & 0 & & & & \end{pmatrix} - \mu I_{2n-2} = O_{2n-2},$$

where I_N and O_N are defined as before. Again A is the Hessian matrix of \tilde{V} and $*$ denotes an element without importance in the computation

of the eigenvalues. The first two eigenvalues are $v \neq 0$ (since $V(\mathbf{s}_0) \neq 0$) and 0. If $\lambda_1, \dots, \lambda_{n-2}$ be the eigenvalues of A , then

$$\mu_i^{1,2} = \frac{1}{4} \{ (b-2)v \pm \sqrt{(2-b)^2 v^2 + 16\lambda_i} \}$$

for $i = 1, \dots, n-2$. Note that, in this case, Lemma 4 implies that A is positive definite, and thus the eigenvalues $\lambda_1, \dots, \lambda_{n-2}$ are all positive. Consequently, for $v > 0$, μ_i^1 is negative and μ_i^2 positive. However, for $v < 0$, μ_i^1 is positive, whereas μ_i^2 is negative. This concludes the proof. \square

We will further state and prove a result that clarifies under what circumstances homothetic solutions exist.

Theorem 5. *A solution of the Manev-type n -body problem is homothetic if and only if the particles form, at all times, a simultaneous central configuration for the potentials V and W .*

Proof. Assume that the solution is homothetic, then $\mathbf{s} \equiv \mathbf{s}_0$, where \mathbf{s}_0 is a constant. Therefore $\mathbf{s}' \equiv 0$ and, from the second of equations (21), $\mathbf{u} \equiv 0$. Thus the homothetic orbits are confined to the invariant plane

$$(27) \quad \mathcal{P} = \{(\rho, \mathbf{s}, v, \mathbf{u}) | \mathbf{s} = \mathbf{s}_0, \mathbf{u} = \mathbf{0}\}.$$

So $\mathbf{u}' = 0$ implies that $\rho^{b-1}[W(\mathbf{s}_0)M\mathbf{s}_0 + \nabla W(\mathbf{s}_0)] + bV(\mathbf{s}_0)M\mathbf{s}_0 + \nabla V(\mathbf{s}_0) = 0$. If ρ is not constant then there are $\rho_1 \neq 0$ and $\rho_2 \neq 0$ with $\rho_1 \neq \rho_2$ such that

$$(28) \quad \begin{aligned} \rho_1^{b-1}[W(\mathbf{s}_0)M\mathbf{s}_0 + \nabla W(\mathbf{s}_0)] &= -[bV(\mathbf{s}_0)M\mathbf{s}_0 + \nabla V(\mathbf{s}_0)] \\ \rho_2^{b-1}[W(\mathbf{s}_0)M\mathbf{s}_0 + \nabla W(\mathbf{s}_0)] &= -[bV(\mathbf{s}_0)M\mathbf{s}_0 + \nabla V(\mathbf{s}_0)]. \end{aligned}$$

This means that $[bV(\mathbf{s}_0)M\mathbf{s}_0 + \nabla V(\mathbf{s}_0)] = 0$ and $[W(\mathbf{s}_0)M\mathbf{s}_0 + \nabla W(\mathbf{s}_0)] = 0$, i.e. that \mathbf{s}_0 is a simultaneous central configuration for the potentials V and W . If ρ is constant, $\rho' = 0$ and either $\rho \equiv 0$ or $v \equiv 0$. The first case is trivial, whereas in the latter case $-\rho^{b-1}W(\mathbf{s}) - bV(\mathbf{s}) = 0$. But this is impossible since $V > 0$ and $W > 0$.

If $\mathbf{s} \equiv \mathbf{s}_0$ is, at all times, a simultaneous central configuration for V and W , then the solution is obviously homothetic. \square

The next result proves the existence and uniqueness of heteroclinic homothetic solutions for $b > 1$.

Theorem 6. *Let \mathbf{s}_0 be a simultaneous central configuration for the potentials V and W . Then, if $b > 1$, every energy surface of negative constant ($h < 0$), contains a unique homothetic solutions defined on $(-\infty, \infty)$, satisfying $\mathbf{s} = \mathbf{s}_0$ for all times and such that $\rho(\tau) \rightarrow 0$*

when $\tau \rightarrow \pm\infty$. In other words the solution begins and ends in a total collapse, maintaining for all times the same central configuration.

Proof. Since \mathbf{s}_0 is a simultaneous central configuration for the potentials V and W , we have that $[bV(\mathbf{s}_0)M\mathbf{s}_0 + \nabla V(\mathbf{s}_0)] = 0$ and $[W(\mathbf{s}_0)M\mathbf{s}_0 + \nabla W(\mathbf{s}_0)] = 0$. Consequently, the set

$$\mathcal{P} = \{(\rho, \mathbf{s}, v, \mathbf{u}) \mid \mathbf{s} = \mathbf{s}_0, \mathbf{u} = \mathbf{0}\}$$

is invariant for the equations of motion. Restricting these equations to \mathcal{P} , we get

$$(29) \quad \begin{aligned} \rho' &= \rho v \\ v' &= \frac{b}{2}v^2 - \rho^{b-1}W(\mathbf{s}_0) - bV(\mathbf{s}_0), \end{aligned}$$

while the energy relation becomes

$$\frac{1}{2}v^2 - \rho^{b-1}W(\mathbf{s}_0) - V(\mathbf{s}_0) = h\rho^b.$$

Equations (29) become

$$\begin{aligned} \rho' &= \rho v \\ v' &= (b-1)\rho^{b-1}W(\mathbf{s}_0) + b\rho^b h. \end{aligned}$$

This leads to

$$\frac{dv}{d\rho} = \frac{1}{v}[(b-1)\rho^{b-2}W(\mathbf{s}_0) + b\rho^{b-1}h],$$

which yields

$$(30) \quad \frac{v^2}{2} = \rho^{b-1}W(\mathbf{s}_0) + \rho^b h + K,$$

where, if $b > 1$, we choose $K = V(\mathbf{s}_0)$. If $b > 1$ and $h \geq 0$, then $|v| \geq \pm\sqrt{2V(\mathbf{s}_0)}$ and the homothetic orbits are not heteroclinic. If $h < 0$, there is a unique curve connecting the points $(\sqrt{2V(\mathbf{s}_0)}, 0)$ and $(-\sqrt{2V(\mathbf{s}_0)}, 0)$ on the plane (v, ρ) . These facts prove the theorem. \square

The following result shows that the above property is also true for the equilateral central configurations.

Corollary 1. *Let \mathbf{s}_0 be an equilateral central configuration for the potential U . Then, if $b > 1$, every energy surface of negative constant ($h < 0$) contains a unique heteroclinic homothetic solution.*

Proof. Clearly \mathbf{s}_0 is a simultaneous central configuration for the potentials V and W . The proof follows from Theorem 6. \square

Two submanifolds \mathcal{E}_1 and \mathcal{E}_2 of a submanifold \mathcal{E} are said to be transverse at a point x if one of the following situation arises:

- (1) $\mathcal{E}_1 \cap \mathcal{E}_2 = \emptyset$;
- (2) $x \in \mathcal{E}_1 \cap \mathcal{E}_2$ and $T_x\mathcal{E}_1 + T_x\mathcal{E}_2 = T_x\mathcal{E}$, where $T_x\mathcal{E}$ denotes the tangent space to \mathcal{E} at the point x .

We can now prove the following result:

Theorem 7. *In the planar Manev-type n -body problem with $b > 2$, a necessary condition for having a transversal homothetic solution $\gamma_h(\mathbf{s}_0)$ in E_h with $h < 0$ is that \tilde{V} be a non-degenerate minimum at the point \mathbf{s}_0 associated with the homothetic solution.*

Proof. Let $\gamma_h(\mathbf{s}_0)$ be a transversal homothetic solution in E_h with $h < 0$. Then by Theorem 3 both $W^u(\mathbf{s}_0^+)$ and $W^s(\mathbf{s}_0^-)$ are $(2n - 2 - \text{ind}(\mathbf{s}_0))$ -dimensional and E_h is $(4n - 5)$ -dimensional. Since $\gamma_h(\mathbf{s}_0) \in W^u(\mathbf{s}_0) \cap W^s(\mathbf{s}_0)$ and $\gamma_h(\mathbf{s}_0)$ is transversal we have that

$$\dim E_h \leq \dim W^u(\mathbf{s}_0) + \dim W^s(\mathbf{s}_0) - 1.$$

That is $4n - 5 \leq 4n - 5 - 2 \text{ind}(\mathbf{s}_0)$. Therefore $\text{ind}(\mathbf{s}_0) = 0$ and the function \tilde{V} has a nondegenerate minimum at \mathbf{s}_0 . \square

8. SIMULTANEOUS CONFIGURATIONS AND RELATIVE EQUILIBRIA

In this closing section, we will show that, for most choices of the masses in the quasihomogeneous 3-body problem, the collinear central configurations of the potential U are not simultaneous relative equilibria for V and W .

Theorem 8. *Let Σ_3 be the set of masses $(m_1, m_2, m_3) \in \mathbb{R}_+^3$ for which the collinear configurations are simultaneous central configurations for the potentials V and W . Then the set Σ_3 is nonempty and nowhere dense in \mathbb{R}_+^3 .*

Proof. Assume the configuration $\mathbf{s}_V(m_1, m_2, m_3)$ is a collinear central configuration for V and $\mathbf{s}_W(m_1, m_2, m_3)$ is a collinear central configuration for W . In [7], Euler found a complicated formula that expresses the ratio of the distances between the masses for any rectilinear central configuration in the Newtonian case. Euler's formula can be directly extended to any homogeneous potential. Moreover, the fact that Euler's expression is an analytic function of the masses remains true in the homogeneous case. Therefore both \mathbf{s}_V and \mathbf{s}_W are analytic functions of m_1, m_2 and m_3 , as long as the masses are positive. Consequently the function $z = \mathbf{s}_V - \mathbf{s}_W$ is also an analytic function of the masses.

For the function V , Euler's formula depends on a , whereas for W it depends on b . So in general $\mathbf{s}_V \neq \mathbf{s}_W$, therefore for every a and b

with $a \neq b$ there are values of the masses for which $z \neq 0$. Since z is a nonzero analytic function, its zeroes form a nowhere dense set.

The nonemptiness of the set of simultaneous central configurations follows from noticing that if $m_1 = m_3$ and the mass m_2 is located halfway between the other two, then the three masses form a simultaneous central configuration for V and W . □

A consequence of Theorems 5 and 8 is that, for most values of the masses, there are no rectilinear homothetic orbits. More precisely:

Corollary 2. *If $(m_1, m_2, m_3) \in \mathbb{R}_+^3 \setminus \Sigma_3$, then there are no rectilinear homothetic orbits.*

This shows that the rectilinear homothetic orbits are characteristic to homogeneous potentials, but they prove unlikely in the quasihomogeneous case.

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