THETA CONSTANTS IDENTITIES FOR JACOBIANS OF CYCLIC 3-SHEETED COVERS OF THE SPHERE AND REPRESENTATIONS OF THE SYMMETRIC GROUP

BY YAACOV KOPELIOVICH

To my friend Elizabeth Drake

Abstract. We find identities between theta constants with rational characteristics evaluated at period matrix of R, a cyclic 3 sheeted cover of the sphere with $3k$ branch points $\lambda_1 ... \lambda_{3k}$. These identities follow from Thomae formula [BR]. This formula expresses powers of theta constants as polynomials in $\lambda_1 ... \lambda_{3k}$. We apply the representation of the symmetric group to find relations between the polynomials and hence between the associated theta constants.

1. Introduction

Let R be a Riemann surface with the equation:

$$y^3 = \prod_{i=1}^{i=3m} (z - \lambda_i)(*).$$

We find relations that are satisfied by theta constants with rational characteristics evaluated at τ_R, the period matrix of R. Special type identities for period matrices are known in the case of a general Riemann surface (Schottky-Jung identities). For hyperelliptic curves there are vanishing theta constants of even characteristics that characterize the associated period matrix. According to Mumford, [Mu] special relations of non vanishing of theta constants evaluated at period matrices of hyperelliptic curves were obtained by Frobenius.

The original Schottky problem seeks special relations among theta constants that characterize the entire moduli space of algebraic curves of genus g. In this note we seek special relations that are satisfied by n-sheeted cyclic covers of the sphere. When $n = 2$ cyclic covers are just hyperelliptic curves. The next case is $n = 3$ and we find relations between theta constants with rational characteristics evaluated at τ_R the period matrices of such curves .

These identities are a result of Thomae formula for cyclic n sheeted covers of the sphere. This formula expresses powers of such theta constants evaluated at the period matrix τ_R through polynomial expression of λ_i. A relation between these polynomials produces a relation between associated theta constants. Applying the representation theory of the symmetric group, S_{3m} we produce a basis for the vector space spanned by the polynomials and as a result relations between the associated theta constants.

For the simplest case of 6 branch points our results overlap with results of Matsumoto [Ma]. In his paper Matsumoto finds the explicit action of S_6 on theta

Date: 04 April 2007.

constants evaluated at τ_R and expresses branch points λ_i as rational functions of theta constants. As a result he writes identities between cubic powers of these constants which essentially coincide with the identities obtained by us in the last section of our note. Using the representation theory of S_6 we see that the space generated by theta constants is 5 dimensional. This seems to be a new result even in this case. We note that the Algebraic dimension of this particular family of curves is 3.

This work was partially done during a visit to the TAMU math department and the author thanks the department for the invitation and kind hospitality. I thank Samuel Grushevsky and Mike Fried for constructive remarks on this note.

2. Thomae formula for cyclic covers and relations between theta constants

We explain the general Thomae formula following [Na] for an algebraic curve R given by the equation:

$$y^3 = \prod_{i=1}^{i=3m} (z - \lambda_i)(*)$$

We denote $f : R \mapsto \mathbb{CP}^1$ the projection $(z, y) \mapsto z$. Define $Q_i = f^{-1}(\lambda_i)$, to be the unique branch point on R that is the pre image of λ_i. Fix a homology basis $a_1, a_2...a_{3m-2}, b_1, b_2, ..., b_{3m-2}$ on R such that the intersections are $a_i a_j = 0 = b_i b_j$ and $a_i b_j = 1$. Let $v_1...v_{3m-2}$ be a basis of standard holomorphic differentials dual to the basis $a_1, a_2...a_{3m-2}, b_1, b_2, ..., b_{3m-2}$ i.e. $\int_{a_i} v_j = 0, \int_{b_i} v_j = \delta_{ij}$. Now fix an ordering of λ_i. Let ϕ be the automorphism of order 3 defined by $(z, y) \mapsto (z, \omega y)$ for $\omega^3 = 1$. We write $\alpha \equiv \beta$ for linear equivalent of divisors, i.e. if there exists a function $g : R \mapsto CP^1$ and $div(g) = \alpha - \beta$. The group $Div^0/\equiv$ is $Jac(R)$, the Jacobian of R. (Div^0 - divisors of degree 0.) Let ψ be the mapping $\psi : Div \mapsto Div/\equiv$. Then the following lemma is true:

Lemma 2.1. *Let $P_1, P_2 \in R, P_1 \neq P_2$ and*

$$D_i = P_i + \phi(P_i) + \phi^2(P_i), i = 1, 2$$

then $\psi(D_1) \equiv \psi(D_2)$.

Proof. Let $f_1(P) = \frac{f(P)-f(P_1)}{f(P)-f(P_2)}$, then $div(f_1) = D_1 - D_2$. □

Define $D = \psi\left(P + \phi(P_i) + \phi^2(P_i)\right)$ as the equivalence class in the Jacobian.

Lemma 2.2. *Let K be the canonical divisor of R Then the following holds:*

(1)
$$D \equiv 3Q_i \equiv \infty_1 + \infty_2 + \infty_3$$

(2)
$$K \equiv (2m - 2)D$$

(3)
$$\sum_1^{3m} Q_i \equiv mD$$

Proof. The first item follows exactly as in the previous lemma. To show the rest, note that $z\frac{dz}{w^2}$ is a holomorphic differential with the divisor Q_1^{6m-6}. □

Now let $\Lambda = \{\Lambda_1, \Lambda_2, \Lambda_3\}$ be a partition of $\{1, 2, 3, 4, 5, ...3m\}$ with $|\Lambda_i| = m$ for $i = 1, 2, 3$. We are interested in the following divisor e_Λ associated with the partition:

$$e_\Lambda = X_{\Lambda_1} + 2X_{\Lambda_2} - D - \Delta$$

where for each subset S of $\{1, 2...3m\}$ we set

$$X_S = \sum_{Q_j \in S} Q_j$$

Fix a point $P_0 \in R$ and let $\Phi_{P_0} : R \to Jac(R)$ be given by $\Phi_{P_0}(P) = \left(\int_{P_0}^{P} v_1 ... \int_{P_0}^{P} v_{3m-2}\right)$.

Definition 2.3. Let $\mathbb{H}_g$ denote the set of $g \times g$ symmetric matrices, τ such that the imaginary part of τ is positive definite. For $\varepsilon, \varepsilon', \in \mathbb{R}^g$ and $\tau \in \mathbb{H}_g$ we denote

$$\theta \left[\begin{array}{c} \varepsilon \\ \varepsilon' \end{array} \right] (\tau) = \sum_{l \varepsilon \mathbb{Z}^{2g}} \exp 2\pi i \left\{ \frac{1}{2} \left(l + \frac{\varepsilon}{2} \right)^t \tau \left(l + \frac{\varepsilon}{2} \right) + \left(l + \frac{\varepsilon}{2} \right)^t \frac{\varepsilon'}{2} \right\}$$

This series is uniformly and absolutely convergent on compact subsets of $\mathbb{C}^g \times \mathbb{H}_g$. To each $w \in \mathbb{C}^{3m-2}$ associate a unique $w_1, w_2 \in \mathbb{R}^g$ such that $w = w_1 + \tau w_2$.

[Na] proves the following formula for theta constants with characteristics associated to divisors e_Λ. see [BR] as well:

Theorem 2.4. *The divisor e_Λ is a point of order 6 on the Jacobian and*

(1) $$\theta[e_\Lambda]^6 (\tau_R) = C_\Lambda (detA)^3 ((\Lambda_0\Lambda_0)(\Lambda_1\Lambda_1)(\Lambda_2\Lambda_2))^3 (\Lambda_0\Lambda_1)(\Lambda_1\Lambda_2)(\Lambda_0\Lambda_2)$$

Here A is the matrix of certain differentials integrated with respect to a_i. and if

$$\Lambda_i = \{i_1 < ... < i_m\}, \Lambda_j = \{j_1 < ... < j_m\}$$

then

$$(\Lambda_i\Lambda_i) = \prod_{k<l} (\lambda_{i_k} - \lambda_{i_l}), \quad (\Lambda_i\Lambda_j) = \prod_{k=1,l=1}^{m} (\lambda_{i_k} - \lambda_{j_l})$$

We apply the theorem to generate special relations between theta functions with characteristics e_Λ, evaluated at τ_R. For each partition Λ denote the polynomial on the right hand side of the last equation by p_Λ. To obtain identities for $\theta[e_\Lambda]$ we search for identities between p_Λ. The key observation that allows us to simplify the problem is the following form of the polynomials: choose $\Lambda = \{\{1, 2..., m\}, \{m + 1, ..., 2m\}, \{2m, ..., 3m\}\}$. Then by definition of p_Λ the factor $\prod_{i=1}^{3} \prod_{j=1}^{3} \Lambda_i\Lambda_j$ is the discriminant and a common factor for each p_Λ which does not depend on the partition Λ. Thus identities between $\theta^6[e_\Lambda]$ are equivalent to identities between the polynomials

$$((\Lambda_0\Lambda_0)(\Lambda_1\Lambda_1)(\Lambda_2\Lambda_2))^2 .$$

Consequently, identities between $\sqrt{\theta^6[e_\Lambda]}$ are equivalent to identities between the polynomials:

$$((\Lambda_0\Lambda_0)(\Lambda_1\Lambda_1)(\Lambda_2\Lambda_2)) .$$

To get a hint for the result observe that the group S_{3m} acts naturally on the polynomials $((\Lambda_0\Lambda_0)(\Lambda_1\Lambda_1)(\Lambda_2\Lambda_2))$ via its action on the partitions of $\{1...3m\}$. Thus $Span(((\Lambda_0\Lambda_0)(\Lambda_1\Lambda_1)(\Lambda_2\Lambda_2))$, is a vector space and has a representation of S_{3m} on it.

3. Explicit Basis

In this section we provide an explicit basis for the space of polynomials from the previous section. We imitate the process described in [J] to construct a basis for the irreducible representation of the symmetric group of S_n. For complex numbers these representations are completely classified. We describe the construction for any representation of the symmetric group and obtain the relevant case of cyclic covers as an immediate corollary of the general case. We remind the reader some facts from the representation theory of S_n.

Let n be a natural number and let $k_1 ... k_m$ be a partition of n. i.e. $\sum_{i=1}^{m} k_i = n$ and $k_1 \geq k_2 \geq k_3 ... \geq k_m$.

Definition 3.1. A Young diagram associated to a partition consists of m rows such that i'th row has k_i elements.

Definition 3.2. Let Y be a Young diagram; a tableau is obtained by distributing the numbers $\{1...n\}$ within the m rows with the following properties

- Each row contains exactly k_i elements
- The numbers in each row form an increasing sequence

Assume that $\Lambda = \{\Lambda_0, ..., \Lambda_k\}$ is a tableau of n. Define the polynomial:

$$(\Lambda_i \Lambda_i) = \prod_{i_k < i_l, \{i_k, i_l\} \in \Lambda_i} (\lambda_{i_k} - \lambda_{i_l}), \quad \text{where} \quad p_\Lambda = \prod_{i=1}^{k} (\Lambda_i \Lambda_i).$$

The symmetric group, S_n acts on Λ and therefore acts on the polynomials p_Λ. To find the basis for p_Λ we use a modification of Garnier relation [J] (7.1) to construct a basis for the polynomials.[1] Arrange the tableau in columns (i.e. the first column will be elements of Λ_1 the second column elements of Λ_2 etc). Overall we have k columns for Λ. Let X be a subset of the $i-th$ column of Λ and Y is a subset of the $i+1-th$ column of Λ. Let $\sigma_1 ... \sigma_k$ be coset representatives for $S_{X \times Y}$ in $S_{X \bigcup Y}$. Then we have the Garnier relations:

Theorem 3.3. *Let μ_i denote the number of elements in the $i-th$ column of Λ. if $|X \bigcup Y| > \mu_i$ then*

$$\sum_{m=1}^{k} sign(\sigma_m) (p_{\sigma_m \Lambda}) = 0.$$

Proof. If $|X \bigcup Y| > \mu_i$, by the pigeon hole principle there exists an involution δ such that $\sigma_m \Lambda$ is invariant under it. Thus

$$\sum_{m=1}^{k} sign(\sigma_m) (p_{\sigma_m \Lambda}) = \sum_{m=1}^{k} sign(\sigma_m) (p_{\delta \sigma_m \Lambda}) =$$

$$= - \sum_{m=1}^{k} sign \sigma_m (p_{\sigma_m \Lambda}) = 0$$

□

In order to exhibit an explicit basis we define a standard Young tableau

[1] We were not able to find a reference to our approach of constructing Specht modules though we are confident its a folklore.

Definition 3.4. A standard tableau is a tableau where the rows and the columns are arranged in an increasing order.

Definition 3.5. We define an ordering on the set of tableaux by setting $\Lambda^1 < \Lambda^2$ if there is an i such that

- if $j > i$ than j is in the same column of Λ^1, Λ^2
- i is in more left column in Λ^1 than Λ^2.

Theorem 3.6. *Let $\Lambda^1 ... \Lambda^k$ be the collection of standard tableaux for a given partition. Then $p_{\Lambda^1} ... p_{\Lambda^k}$ is a basis for the vector space spanned by Λ.*

Proof. We follow [J] in the proof. We show that p_{Λ^k} spans any other polynomial corresponding to our partition. Let t be a tableau and suppose by induction that the theorem is proved for each t_1 tableau such that $t_1 < t$. If t is non standard there exists adjacent columns $a_1 < ... < a_q < ... < a_r$ and $b_1 < b_2 ... < b_q < ... b_s$ such that $a_q > b_q$. Apply Garnier relation for $X = a_1 ... a_r, Y = b_1 ... b_q$. For each σ a representative in $S_{X \bigcup Y}$ in $S_{X \times Y}$ we have that $[t\sigma] < t$ by the definition of the order $<$. The result follows immediately from the induction hypothesis. □

Definition 3.7. For an element k of the tableau t Let C_k, R_k be the unique column and row k belongs to. The hook of k, h_k is the number of elements beneath k in C_k plus the number of elements to the right of k in R_k (include the element itself in the row but not in the column.)

It is well known that the number of standard tableaux equals to

$$\frac{n!}{\prod_k h_k} \tag{2}$$

See [J].

4. The ideal of theta identities

We apply the theory of the previous paragraph to cyclic covers of order 3. According to the theory, the hooks of the partitions correspond to tableau with 3 rows and m elements in each row. Our first corollary is

Corollary 4.1. *The dimension of the polynomials p_Λ (and hence the vector space spanned by $\sqrt{\theta^6[e_\Lambda](\tau_R)}$ corresponding to them) is: $\frac{(3m)! \times 2}{(m+2)!(m+1)!m!}$.*

Hence we can also give a basis for $\theta^6[e_\Lambda](\tau_R)$ that correspond to the different partitions e_Λ.

Corollary 4.2. *The set of $\sqrt{\theta^6[e_{\Lambda_S}](\tau_R)}$, Λ_S is a standard partition is a basis for a vector space spanned by $\sqrt{\theta^6[e_\Lambda](\tau_R)}$. In particular each $\sqrt{\theta^6[e_\Lambda](\tau_R)}$, Λ can be written as a linear combination of elements from the set $\sqrt{\theta^6[e_{\Lambda_S}](\tau_R)}$.*

5. Example

Let us revisit the case when there are 6 branch points and the genus of the surface is 4. In this case, by the formula for the dimension, the number of basis functions, $\theta^3[e_\Lambda]$ is : $2 \times \frac{6!}{4!3!2!} = 5$. We enumerate the 15 partitions as well as the the polynomials that correspond to them:

(1) $\Lambda = \{(1,2),(3,4),(5,6)\}\, p_\Lambda = (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)(\lambda_5 - \lambda_6)$
(2) $\Lambda = \{(1,2),(3,5),(4,6)\}\, p_\Lambda = (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_5)(\lambda_4 - \lambda_6)$

(3) $\Lambda = \{(1,2),(3,6),(4,5)\}\, p_\Lambda = (\lambda_1-\lambda_2)(\lambda_3-\lambda_6)(\lambda_4-\lambda_5)$
(4) $\Lambda = \{(1,3),(2,4),(5,6)\}\, p_\Lambda = (\lambda_1-\lambda_3)(\lambda_2-\lambda_4)(\lambda_5-\lambda_6)$
(5) $\Lambda = \{(1,3),(2,5),(4,6)\}\, p_\Lambda = (\lambda_1-\lambda_3)(\lambda_2-\lambda_5)(\lambda_4-\lambda_6)$
(6) $\Lambda = \{(1,3),(2,6),(4,5)\}\, p_\Lambda = (\lambda_1-\lambda_3)(\lambda_2-\lambda_6)(\lambda_4-\lambda_5)$
(7) $\Lambda = \{(1,4),(2,5),(3,6)\}\, p_\Lambda = (\lambda_1-\lambda_4)(\lambda_2-\lambda_5)(\lambda_3-\lambda_6)$
(8) $\Lambda = \{(1,4),(2,6),(3,5)\}\, p_\Lambda = (\lambda_1-\lambda_4)(\lambda_2-\lambda_6)(\lambda_3-\lambda_5)$
(9) $\Lambda = \{(1,4),(2,3),(5,6)\}\, p_\Lambda = (\lambda_1-\lambda_4)(\lambda_2-\lambda_3)(\lambda_5-\lambda_6)$
(10) $\Lambda = \{(1,5),(2,3),(4,6)\}\, p_\Lambda = (\lambda_1-\lambda_5)(\lambda_2-\lambda_3)(\lambda_4-\lambda_6)$
(11) $\Lambda = \{(1,5),(2,4),(3,6)\}\, p_\Lambda = (\lambda_1-\lambda_5)(\lambda_2-\lambda_4)(\lambda_3-\lambda_6)$
(12) $\Lambda = \{(1,5),(2,6),(3,4)\}\, p_\Lambda = (\lambda_1-\lambda_5)(\lambda_2-\lambda_6)(\lambda_3-\lambda_4)$
(13) $\Lambda = \{(1,6),(2,3),(4,5)\}\, p_\Lambda = (\lambda_1-\lambda_6)(\lambda_2-\lambda_3)(\lambda_4-\lambda_5)$
(14) $\Lambda = \{(1,6),(2,4),(3,5)\}\, p_\Lambda = (\lambda_1-\lambda_6)(\lambda_2-\lambda_4)(\lambda_3-\lambda_5)$
(15) $\Lambda = \{(1,6),(2,5),(3,4)\}\, p_\Lambda = (\lambda_1-\lambda_6)(\lambda_2-\lambda_5)(\lambda_3-\lambda_4)$

The basis for the vector space of the polynomials corresponds to the following standard tableaux:

(1) $\Lambda = \{(1,2),(3,4),(5,6)\}\, p_\Lambda = (\lambda_1-\lambda_2)(\lambda_3-\lambda_4)(\lambda_5-\lambda_6)$
(2) $\Lambda = \{(1,2),(3,5),(4,6)\}\, p_\Lambda = (\lambda_1-\lambda_2)(\lambda_3-\lambda_5)(\lambda_4-\lambda_6)$
(3) $\Lambda = \{(1,3),(2,4),(5,6)\}\, p_\Lambda = (\lambda_1-\lambda_3)(\lambda_2-\lambda_4)(\lambda_5-\lambda_6)$
(4) $\Lambda = \{(1,3),(2,5),(4,6)\}\, p_\Lambda = (\lambda_1-\lambda_3)(\lambda_2-\lambda_5)(\lambda_4-\lambda_6)$
(5) $\Lambda = \{(1,4),(2,5),(3,6)\}\, p_\Lambda = (\lambda_1-\lambda_4)(\lambda_2-\lambda_5)(\lambda_3-\lambda_6)$

The rest of the 10 polynomials can be rewritten as a linear combination of the set above applying Garnier's algorithm as in Theorem 3.7. For example we have:

$$(\lambda_1-\lambda_2)(\lambda_3-\lambda_6)(\lambda_4-\lambda_5) = -(\lambda_1-\lambda_2)(\lambda_3-\lambda_4)(\lambda_5-\lambda_6)+(\lambda_1-\lambda_2)(\lambda_3-\lambda_5)(\lambda_4-\lambda_6)$$

$$(\lambda_1-\lambda_3)(\lambda_2-\lambda_6)(\lambda_4-\lambda_5) = -(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)(\lambda_5-\lambda_6)+(\lambda_1-\lambda_3)(\lambda_2-\lambda_5)(\lambda_4-\lambda_6)$$

$$(\lambda_1-\lambda_6)(\lambda_2-\lambda_5)(\lambda_3-\lambda_4) = (\lambda_1-\lambda_4)(\lambda_2-\lambda_5)(\lambda_3-\lambda_6)-(\lambda_1-\lambda_3)(\lambda_2-\lambda_5)(\lambda_4-\lambda_6)$$

The others polynomials can be expressed in a similar way leading to identities between $\sqrt{\theta^6[e_\Lambda]\,(\tau_R)}$ in this case. Let us conclude with the following remarks on the identities above: In the hyperelliptic curve case the identities between integral characteristics of theta functions evaluated at period matrix of hyperelliptic curves arise from vanishing properties of theta functions. In our case it is interesting to investigate whether an analogous situation can arise. The only source of cubic theta identities known to the author, is the following theorem in [Ko]:

Theorem 5.1. *Let* $\begin{bmatrix} \mu \\ \mu' \end{bmatrix}$ *be an odd integral theta characteristics in genus* $3m-2$ *Then for any* $\tau \in \mathbb{H}_{3m-2}$*:*

$$\sum_{0\le\nu_i\le 3} (-1)^{\sum_{i=1}^{3m-2}\mu_i\nu_i}\theta^3 \begin{bmatrix} \mu \\ \mu' + \frac{2\nu}{3} \end{bmatrix}(0,\tau) = 0 \tag{3}$$

It is plausible that the vanishing of theta constants with rational characteristics of order 3 on τ_R will produce a new proof for the special identities obtained in this note using Thomae formula. Finally note that for all the identities (4) the coefficients are ± 1 It is plausible that this a general phenomenon.

6. CONCLUSION

There exists an extensive literature on Schottky-Jung identities and on theta constants for hyperelliptic curves. In this note we obtained special identities for other classes of algebraic curves. In subsequent notes we plan to pursue and develop further the themes touched in this note, especially applications of similar methods to general Hurwitz spaces and their mapping class groups.

References

[AK] R.Adin, Y.Kopeliovich, Short Eigenvectors and Multidimensional Theta Functions, *Linear Algebra and Appl.* **257**(1)(1997) 49-63

[BR] M. Bershadsky, A. Radul, Fermionic fields on Z_n curves *Comm. in Mathematical Phys.* **116**(4)(1988) 689-700

[FK1] H. Farkas, Y. Kopeliovich, New Theta Constant Identities *Israel Journal of Mathematics* **82**(1)(1993) 133-140

[FK2] H. Farkas, Y.Kopeliovich, New Theta Constant Identities II *Proceeding of AMS.***123**(4)(1995) 1009-1020

[J] G.D.James *The representation theory of the Symmetric Groups* Lecture Notes in Math. vol. 682 (Springer Verlag 1978)

[Ko] Y. Kopeliovich, Multi Dimensional Theta Constant Identities *Journal of Geometric Analysis* **8** (4)(1998) 571-581

[Ma] K.Matsumoto *Theta constants associated with the cyclic triple coverings of the complex projective line branching at six points* Publ. Res. Inst. Math. Sci. **37** (3) (2001) 419-440

[Mu] D. Mumford, *Tata Lectures on Theta II* (Progress in Mathematics, Birkhauser 1984)

[Na] A. Nakayashiki, On the Thomae formula for Z_N curves *Publ. Res. Inst. Math Sci.* **33** (6)(1997) 987-1015

[Th] J. Thomae, Beitrag zur Bestimmung $\theta(0, 0..., 0)$ durch di klassenmoduln algebraicer Funktionen *Crelle's Journal* **71**(1870) 201-222

5736 Las Virgenes Rd. Calabasas CA 91302