# Math 194, problem set #8

For discussion Tuesday December 2

You are *strongly urged* to write up and hand in a careful and complete solution to (at least) one of these problems.

1. Find the least number $A$ such that for any two squares of combined area 1, a rectangle of area $A$ exists such that the two squares can be packed in the rectangle (without interior overlap). You may assume that the sides of the squares are parallel to the sides of the rectangle. (Putnam 1996)

2. Let $C_1$ and $C_2$ be circles of radius 1, tangent to each other and to the $x$-axis. Construct a sequence of circles $C_n$ such that $C_{n+1}$ is tangent to $C_n$, $C_{n-1}$, and the $x$-axis. What is the radius $r_n$ of $C_n$? (Bankoff)

3. The radius of the inscribed circle of a triangle is 4, and the segments into which one side is divided by the point of contact have length 6 and 8. Determine the lengths of the other two sides. (Larson 8.1.13)

4. On the sides of an arbitrary parallelogram, squares are constructed lying exterior to it. Prove that their centers form the vertices of a square.

   (Larson 8.3.12)

5. The *midset* of the point sets $S$ and $T$ is the set of midpoints of all the line segments $XY$, where $X$ is in $S$ and $Y$ is in $T$.

   (a) If $S$ and $T$ are the perpendicular sides of a $3-4-5$ right triangle, what is the midset of $S$ and $T$?

   (b) If $S$ and $T$ are the skew face diagonals on the opposing faces of a cube, what is the midset of $S$ and $T$? (Konhauser, Velleman, Wagon)

6. (a) Suppose that a regular octagon is tiled with non-overlapping parallelograms. Prove that at least 2 of these parallelograms are rectangles.

(b) Using this method, what can you say about the number of rectangles in a regular 400-gon tiled by parallelograms?

7. Four points are chosen at random on the surface of the sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points? (It is understood that each point is independently chosen relative to a uniform distribution on the sphere.) (Putnam 1992)

8. Let $T$ be an acute triangle. Inscribe a rectangle $R$ in $T$ with one side along a side of $T$. Then inscribe a rectangle $S$ in the triangle formed by the side of $R$ opposite the side on the boundary of $T$, and the other two sides of $T$, with one side along the side of $R$. For any polygon $X$, let $A(X)$ denote the area of $X$. Find the maximum value, or show that no maximum exists, of $\frac{A(R)+A(S)}{A(T)}$, where $T$ ranges over all triangles and $R, S$ over all rectangles as above.
(Putnam 1985)

9. Calculate the volume of a tetrahedron $ABCD$ where $AB = AC = AD = 5$ and $BC = 3$, $CD = 4$, and $BD = 5$. (Barbeau, Klamkin, Moser)

10. Suppose you have a polygon of perimeter 12, whose vertices are all on lattice points, whose sides all have integer lengths, and whose area, $A$, is an integer. Show that $A$ can be $3, 4, 5, 6, 7, 8$ or $9$.