Math 230B final

Due: March 23, 2005, 4:00pm

Turn in your completed exam to Jennifer Dugan in MSTB 103.

You may use your class notes and the portions of the text that were covered in class. No other sources are permitted. No collaboration is permitted. *Justify your answers carefully and completely.*

There are 8 problems on 2 pages, worth a total of 100 points. If a question is not clear, please e-mail me (`krubin@math.uci.edu`) and ask.

(9 points) 1. Suppose E and F are fields, and $\varphi : E \to F$ is a (nonzero) field homomorphism. Prove that the characteristic of E is equal to the characteristic of F.

(12 points) 2. Suppose $a \in \mathbf{Z}$, $a \neq 2$. Let $f(x) = x^3 + ax^2 + (a-3)x - 1 \in \mathbf{Q}[x]$.

(a) Show that $f(x)$ is irreducible.
(b) Show that if $\alpha \in \mathbf{C}$ is a root of $f(x)$, then so is $-1 - 1/\alpha$.
(c) If $\alpha \in \mathbf{C}$ is a root of $f(x)$, show that $\mathbf{Q}(\alpha)$ is Galois over $\mathbf{Q}$ and describe $\mathrm{Gal}(\mathbf{Q}(\alpha)/\mathbf{Q})$.

(15 points) 3. Each of the following examples gives a ring R and an ideal I of R. In each case state (and *justify*)

- whether or not I is a prime ideal, and
- whether or not I is a maximal ideal.

(a) $R = \mathbf{Z} \times \mathbf{Z}$, I is the principal ideal generated by $(1, 5)$.
(b) $R = \mathbf{Z} \times \mathbf{Z}$, $I = 7\mathbf{Z} \times 7\mathbf{Z}$.
(c) $R = \mathbf{Z}[x]$, I is the principal ideal $(x^2 - 1)$.
(d) $R = \mathbf{Z}[x]/x^2\mathbf{Z}[x]$, I is the principal ideal (x).
(e) $R = \mathbf{Z}[i]$ (where $i = \sqrt{-1}$), I is the principal ideal (7).
(f) $R = \mathbf{Q}[x, y]$, I is the principal ideal $(x - 2y)$.

(12 points) 4. Let G be a group of order 380. Prove that G is not simple.

(12 points) 5. Let $\mathbf{R}$ denote the real numbers, and let $k = \mathbf{R}(t)$ (the field of rational functions in the variable t). Describe explicitly the Galois group of the splitting field of the polynomial $f(x) = x^4 + t \in k[x]$. (I.e., describe the automorphisms and identify the group.)

(12 points) 6. (a) Suppose S is a commutative ring, $I \subsetneq S$ is an ideal, and M is an S/I-module. Show that M is an S-module (in a natural way).

(b) Describe all finitely generated modules over the ring $R = \mathbf{C}[x]/(x^3)$. (Note that R is not a domain, so it is not a PID.)

(16 points) 7. Suppose that R is a commutative ring. We say that $a \mid b$ (or "a divides b") if there is an element $c \in R$ such that $b = ac$. We say that R is a *valuation ring* if for every pair of elements $a, b \in R$ either $a \mid b$ or $b \mid a$.

Suppose that R is a valuation ring.

(a) If I and J are ideals of R, show that either $I \subseteq J$ or $J \subseteq I$.

(b) If I is a finitely generated ideal of R, show that I is principal.

(c) Show that if R is noetherian, then it has a unique maximal ideal.

(d) Suppose R is noetherian. Show that there is an element $t \in R$ such that every nonzero ideal I of R is generated by t^n for some $n \geq 0$.

(12 points) 8. Suppose A is an 8×8-matrix with entries in $\mathbf{C}$, such that:

- $\dim_{\mathbf{C}}(\ker(2I - A)) = 2$,
- $\dim_{\mathbf{C}}(\ker(2I - A)^2) = 3$,
- $\dim_{\mathbf{C}}(\ker(3I - A)) = 2$,
- $\dim_{\mathbf{C}}(\ker(3I - A)^2) = 4$,
- $\dim_{\mathbf{C}}(\ker(3I - A)^3) = 5$,

(a) What is the characteristic polynomial of A?

(b) What is the minimal polynomial of A?

(c) What is the Jordan normal form of A?

(d) What is the rational canonical form of A?