Proceedings of 4th SCA 74-81 (2007) ?–?

Variational Principles and Large Scale Effective Quantities

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Large scale quantities are essential to multiscale modeling in various areas of science and engineering, and are often associated with variational principles. We illustrate by two examples the role of variational principles in bridging the information between small and large scales. The first example is the variational principle of Kolmogorov-Petrovsky-Piskunov (KPP) front speeds in temporally random shear flows inside an infinite cylinder. The variational principle allows us to bound and compute the front speeds. The second example is a sound amplification scheme for hearing aids based on minimization of perceptually corrected mechanical output of normal and impaired ear models. The two scale approach incorporating mechanics and perception recovers empirical results of hearing aid fitting, and offers adaptivity when input sounds are noisy.

Keywords: Variational Principles, Computation of Large Scale Quantities.

AMS Subject classification: 35K57, 41A60, 65D99.

1. Introduction

Large scale quantities are very helpful for analysis and computation of multiscale problems. Variational principles can be used to either characterize the large scale quantities or include them in a multiscale framework for efficient computation and modeling. We demonstrate this interesting role of variational principles here with two concrete examples arising in dissimilar applications.

The first example is motivated by the study of propagation velocity of premixed turbulent flames or a more general problem of reaction-diffusion front propagation in random media ([5,15,20,21,23,24,26–28],[6,18],[25] and references). A fundamental large scale quantity is the large time front speed that depends on statistics of the random media in a highly nontrivial manner. In combustion literature, ad hoc and formal procedures abound for approximation, such as closures and renormalization group methods [23,28]. We show the recently established variational principles of propagation speeds of KPP reaction-diffusion fronts through temporally random shear flows inside an infinite cylinder [22]. Thanks to the variational principe, we are able to compute the statistical properties of front speeds with both accuracy and ease in contrast to the laborious direct simulations. The KPP equation plays a similar role for the random fronts as Burgers equation for fluid turbulence [7].

The second example concerns with amplifying sounds for people with hearing loss. The problem has been mostly studied empirically ([8,11,4] and references therein). In NAL-NL1 (National Acoustic Labs, Nonlinear Version 1) strategy [4,19], a speech intelligibility (SI) function is defined over a number of frequency bands based on empirical functions (effective and desensitized audibilities) and hearing loss audiogram. The input sound is amplified to maximize SI while keeping loudness no more than what is perceived by a normal ear. It is however not known how to extend the empirical approach if there is noisy interference as is common in natural environment. We present a two scale framework which allows noisy input signals and adaptive amplification to be computed. The two scales are mechanical and perceptual scales respectively. The amplification is a result of minimizing the difference between normal and impaired mechanical ear models subject to a perceptual (large scale) weighting. The perceptual weighting sets the maximal achievable target of amplification which is realistic for severely impaired hearing.

The two examples are discussed in sections two and three. The conclusions, future works, and acknowledgements are in sections four and five.

2. KPP Speeds in Time Random Media

Consider the model equation:

$$u_t = \frac{1}{2}\Delta_z u + B \cdot \nabla_z u + f(u), \tag{2.1}$$

where $u = u(z,t)$, $z = (x,y) \in R \times \Omega$, Ω a bounded open subset of R^{n-1} with Lipschitz continuous boundary, $n \geq 2$; $B = (b(y,t), 0, \ldots, 0)$, $b(y,t)$ is a stationary Gaussian process in t, with a given deterministic profile in y. The nonlinear function $f(u) \in C^1([0,1])$ is a KPP nonlinearity: $f(u) > 0$ for $u \in (0,1)$, $f(0) = f(1) = 0$, $f'(0) = \sup_{u \in (0,1)} f(u)/u$. An example is $f(u) = u(1-u)$.

For compactly supported initial data bounded between 0 and 1, solutions of (2.1) develop into propagating fronts separating the cylindrical domain into a region where $u \approx 1$ and the rest where $u \approx 0$, which correspond to burned (hot) and unburned (cold) states in combustion. For temporally random shear flows, it is more efficient to study front solutions asymptotically without constructing exact traveling fronts. This line of work goes back to Freidlin [9] where variational principles of KPP front speeds in spatially periodic media are obtained by combining the large deviation techniques and Feynman-Kac representation formulas of KPP solutions. This approach has been recently further developed [22] to treat the temporally random shear flows which generate more complexities in path integrals and unbounded variations in time.

Let us make precise our assumptions on the shear field. The function $b(y,t) = b(y,t,\hat{\omega})$ is a mean zero Gaussian random field over (y,t), periodic in y with period L for each fixed t, and stationary in t for each fixed y. The field b is defined over probability space $(\hat{\Omega}, \hat{\mathcal{F}}, Q)$ and has covariance function $\Gamma(y_1, y_2, t_1, t_2) = E_Q[b(y_1, t_t) b(y_2, t_2)]$. The following assumptions hold on $b(y,t)$:

A1: (Periodicity in y) Let $C_P^{0,1}(D)$ denote the space of Lipschitz continuous functions that are periodic on the period cell $D = [0,L]^{n-1}$. For each $\hat{\omega} \in \hat{\Omega}$, there is a continuous map $J(\cdot, \hat{\omega}) : [0, +\infty) \to C_P^{0,1}(D)$ such that $b(\cdot, t, \hat{\omega}) = J(t, \hat{\omega})$.

A2: (Stationarity in t) For each $s \in R^+$ there is a measure preserving transformation $\tau_s : \hat{\Omega} \to \hat{\Omega}$ such that $b(y, \cdot + s, \hat{\omega}) = b(y, \cdot, \tau_s \hat{\omega})$. Hence, Γ depends only on y_1, y_2 and $|t - s|$.

A3: (Ergodicity) The transformation τ_s is ergodic: if a set $A \in \hat{\mathcal{F}}$ is invariant under the transformation τ_s, then either $Q(A) = 0$ or $Q(A) = 1$.

A4: The field b is mean zero, almost surely continuous in (y,t), and has uniformly bounded variance:

$$E[b(y,t)] = 0 \quad E[b(y,t)^2] \le \sigma^2 \quad \text{for all} \;\; y \in D, t \ge 0. \tag{2.2}$$

A5: The function $\hat{\Gamma}(r) = \sup_{y_1,y_2} \Gamma(y_1,y_2,0,r)$ is integrable over $[0,\infty)$:

$$\int_0^\infty \hat{\Gamma}(r)\,dr = p_1 < \infty \tag{2.3}$$

for some finite constant $p_1 > 0$.

A6: There is a finite constant $p_2 > 0$ such that

$$|\Gamma(y_1,y_2,s,t) - \Gamma(y_1,y_3,s,t)| \le p_2|y_3 - y_2|\hat{\Gamma}(|s-t|).$$

For example, a field satisfying Assumptions A1-A6 might have the form $b(y,t) = \sum_{j=1}^N b_1^j(y) b_2^j(t)$, where the functions $b_1^j(y)$ are deterministic, Lipschitz continuous and periodic over D, and the functions $b_2^j(t)$ are mean zero, stationary Gaussian fields in t.

To state the main results, let us define the family of Markov processes associated with the linear part of the operator in (2.1). For a fixed $\hat{\omega} \in \hat{\Omega}$ and for each $z \in R^n$, $t \ge 0$, let $Z^{z,t}(s) = (X^{z,t}(s), Y^{z,t}(s)) \in R^n$ solve the Itô equation:

$$\begin{aligned} dX^{z,t}(s) &= dW_1(s) + b(W_2(s), t-s)\,ds \\ dY^{z,t}(s) &= dW_2(s) \end{aligned} \tag{2.4}$$

with initial condition $Z^{z,t}(0) = z = (x,y) \in R^n$, $W(s) = (W_1(s), W_2(s)) \in R^n$ is the n-dimensional Wiener process with $W(0) = 0$. Let $P^{z,t}$ denote the corresponding family of measures on $C([0,t]; R^n)$. The KPP front speed depends on large deviations of the random variable

$$\eta_z^t(\kappa t) = \frac{x - X^{z,t}(\kappa t)}{\kappa t} \tag{2.5}$$

which is the first component of the average speed of a trajectory over time interval $[0,\kappa t]$, $\kappa \in (0,1]$. We consider only the first component since we are concerned only with propagation in the x direction. The need for the parameter κ results from analyzing the time dependence of the field $b(y,t)$.

Now we state the main results. First, the following lemma allows us to characterize the speed of propagation:

Lemma 1. Assume that A1-A6 hold for the process $b(y,t)$. Then for each $\lambda \in R$, the limit

$$\mu(\lambda, z) = \mu(\lambda) = f'(0) + \lim_{t\to\infty} \frac{1}{t} \log E\left[e^{-\lambda X^{z,t}(t)}\right] \tag{2.6}$$

exists and is a finite constant, almost surely with respect to measure Q and independent of $z \in R^n$. Moreover, $\mu(\lambda)$ is a convex, positive, and even function of λ. Also, $\mu(\lambda)/|\lambda| \to +\infty$ as $|\lambda| \to \infty$.

Let $H(c)$ be the Legendre transform of $\mu(\lambda)$

$$H(c) = \sup_{\lambda\in R}[c \cdot \lambda - \mu(\lambda)],$$

then the speed of propagation can be bounded above in terms of H.

Theorem 2.1 Upper bound on front speed. Let $b(y,t,\hat{\omega})$ satisfy assumptions A1 - A6. Let $u(x,y,t,\hat{\omega})$ solve (2.1) with initial condition $u(x,y,0,\hat{\omega}) = u_0(x)$, where $u_0(x) \in [0,1]$ has compact support and is independent of $\hat{\omega}$. Then, for any closed set $F \subset \{c \in R|\ H(c) > 0\}$,

$$\lim_{t\to\infty} \sup_{y\in D} u(ct, y, t, \hat{\omega}) = 0$$

uniformly in $c \in F$, for almost every $\hat{\omega} \in \hat{\Omega}$.

Therefore, if we define the constant $c^* > 0$ by the variational formula

$$c^* = \inf_{\lambda>0} \frac{\mu(\lambda)}{\lambda}, \tag{2.7}$$

we see from the definition of H that the front spreads asymptotically with speed no greater than c^* in the positive x direction and with speed no greater than c^* in the negative x direction. Although the solution u depends on $\hat{\omega} \in \hat{\Omega}$ since B is a random variable over $\hat{\Omega}$, the function $H(c)$ and the speed c^* are independent of $\hat{\omega}$. They are almost surely constant with respect to $\hat{Q}$, a consequence of the ergodicity assumption A3.

The constant c^* is also almost surely a lower bound on the speed:

Theorem 2.2 Lower bound on front speed. Let $b(y,t,\hat{\omega})$ satisfy assumptions A1 - A6. Let $u(x,y,t,\hat{\omega})$ solve (2.1) with initial condition

$u(x, y, 0, \hat{\omega}) = u_0(x)$, where $u_0(x) \in [0, 1]$ has compact support and is independent of $\hat{\omega}$. Then, for any compact set $K \subset \{c \in R | \; H(c) < 0\}$,

$$\lim_{t \to \infty} \inf_{y \in D} u(ct, y, t, \hat{\omega}) = 1$$

uniformly in $c \in K$, for almost every $\hat{\omega} \in \hat{\Omega}$.

The proofs make use of large deviation estimates of the stochastic flows (2.4), the path integral representation of solutions to equation (2.1), subadditive ergodic theorem, and the Borell inequality for Gaussian fields [1,2,14]. The variational formula (2.7) also provides a numerical method on the front speed c^*. First compute the auxiliary linear initial value problem (periodic in y):

$$\phi_t \;=\; \frac{1}{2}\,\Delta\phi - \lambda\, b(y,t)\,\phi, \quad \phi(y,0) \;\equiv\; 1, \tag{2.8}$$

by for example the Crank-Nicholson scheme. The solution of the linear problem (2.8) corresponds to the formula (2.6). Then at a long enough time t, approximate

$$\mu(\lambda) \approx \mu_t(\lambda) = \lambda^2/2 + f'(0) + \frac{1}{t}\log(\|\phi\|_1). \tag{2.9}$$

The μ_t converges to μ almost surely as t becomes large, so we need only generate one realization of the process b. One may also generate a few realizations and average the μ_t over them. Finally, c^* is obtained by minimization. We refer to [22] for detailed results on the enhancement of c^* and its dependence on the shear correlation length. In short, the variational principle (2.7) reduces the amount of computation significantly and is particularly helpful in simulating large scale front speed c^* in random media. The variational principle (2.7) depends only on the linear part of the reaction-advection-diffusion equation (2.1), implying that a global linearization is at work, reminiscent of the Hopf-Cole transform of Burgers equation.

3. Two Scale Sound Amplifying Scheme

Sound processing in human auditory system follows a multiscale pathway, from mechanics in the inner ear to neural responses in the brain then to perception [10]. Much less is known about the neural part than either the mechanical or the perceptual. An efficient approach would be to lump what is known about the neural part into the mechanical part where Newton's laws hold, then correct the output of this lumped system by perceptual observations. Such a lumped system is formulated and computed recently in [12] to recover nonlinear hearing phenomena such as tone and tone, or tone and noise interactions. The nonlinearity comes from outer hair cell (OHC) structures in the inner ear. Hearing loss is often caused by the weakening of the OHC nonlinearity [17]. By varying the strength of OHC nonlinearity, we can model both normal and impaired ear responses.

Consider tonal (sinusoids) input for simplicity. Let $R_1(A, f)$ be the normal ear response to a sound of amplitude A and frequency f, and $R_2(A, f)$ be the impaired response to the same sound. Then we seek a gain G on the amplitude A such that the distance between $R_1(A, f)$ and $R_2(A * G, f)$ is minimized. The gain G depends on both A and f in general. It turns out that this amplifying scheme is fine in the mid or low frequency range (below 4 kilo-Hertz) but provides too much gain in the high (above 4 kilo-Hertz) range, as compared with data from NAL-NL1 [19] in case of clean signals. The mismatch is for a good reason. For high frequency, especially the severe loss case (over 60 decibel), more gain does not help, in fact reduces the understanding [3,4]. This motivates a perceptual weighting $w = w(A, f)$ to modify the objective function of minimization to the distance of $w(A, f)\, R_1(A, f)$ and $R_2(A * G, f)$. The weighting is most effective in the high frequency region. Such a two scale amplifying scheme is carried out in [13] recently. It recovers NAL-NL1 data for clean signals at various hearing loss levels, and also allows an adaptive amplification for noisy signals. In the noisy regime, signal to noise ratios (SNR) can be estimated from the normal ear model, then the amplification G depends also on the SNR. At low SNR, the amplification is smaller. We refer to [12,13] for details of the ear models, the weighting function w, and numerical results.

4. Conclusions and Future Works

Variational principles are useful tools for representing or incorporating large scale quantities in multi-scale problems for both analysis and computation. In the first example, it remains to establish variational formulas for front speeds in more general (non-shear) random flows, and carry out related computations. In the second example, it remains to study quantitatively how the adaptive (SNR dependent) minimization improves the understanding of the hearing impaired in noisy environment.

5. Acknowledgements

The work was partially supported by NSF grants ITR-0219004, SCREMS-0322962, DMS-0506766. The paper is based on the author's lecture at the 4th International Workshop on Scientific Computing and Applications at Jiaoda, Shanghai, in June of 2005. The author wishes to thank the workshop organizers for their invitation and hospitalities.

References

[1] R. Adler, "An Introduction to Continuity, Extrema and Related Topics for General Gaussian Processes", Institute of Math Stat, Lecture Notes-Monograph Series, Vol. 12, 1990.

[2] M.A. Akcoglu and U. Krengel, *Ergodic theorems for superadditive processes*, J. Reine Angew Math. Vol. 323 (1981), pp. 53-67.

[3] T. Ching, H. Dillon, and D. Byrne, *Speech recognition of hearing-impaired listeners: Predictions from audibility and the limited role of high-frequency amplification.* J. Acousti. Soc. America, 103:1128-1140, 1998.

[4] T. Ching, H. Dillon, R. Katsch, and D. Byrne, *Maximizing Effective Audibility in Hearing Aid Fitting.* Ear and Hearing, 22.3:212-224, 2001.

[5] P. Clavin, and F. A. Williams, *Theory of premixed-flame propagation in large-scale turbulence*, J. Fluid Mech., 90, (1979), pp. 598-604.

[6] J. Conlon, C. Doering, *On Traveling Waves for the Stochastic FKKP Equation*, Jour. Stat Physics, Vol. 120, Nos. 3-4, pp 421-477, 2005.

[7] Weinan E, Y. Sinai, *New results in mathematical and statistical hydrodynamics*, Russian Math Surveys, 55(2000), no. 4, pp 635-666.

[8] H. Dillon, "Hearing Aid", Boomerang Press, Sydney, 2001.

[9] M.I. Freidlin, "Functional Integration and Partial Differential Equations", Ann. Math. Stud. 109, Princeton University Press, Princeton, NJ, 1985.

[10] C.D. Geisler, "From Sound to Synapse", Oxford University Press, Oxford, 1998.

[11] M. Killion, *Talking Hair Cells: What They Have to Say About Hearing Aids*, in "Hair Cells and Hearing Aids", 1996, C. Berlin eds, Singular Publishing Group Inc.

[12] Y. Kim and J. Xin, *A two-dimensional nonlinear nonlocal feed-forward cochlear model and time domain computation of multitone interactions.* SIAM J. Multiscale Modeling and Simulations, Vol. 4, No. 2, pp 664-690, 2005.

[13] Y. Kim, J. Xin, Y. Qi, *A Study of Hearing Aid Gain Functions Based on a Nonlinear Nonlocal Feedforward Cochlea Model*, Hearing Research, Volume 215, Issues 1-2, May 2006, pp 84-96.

[14] J. P. C. Kingman, *The Ergodic Theory of Subadditive Stochastic Processes*, Journal of the Royal Statistical Society, Series B, Vol. 30, No. 3, (1968), pp. 499-510.

[15] A. Majda and P.E. Souganidis, *Flame fronts in a turbulent combustion model with fractal velocity fields*, Comm Pure Appl Math, Vol. LI (1998), pp. 1337-1348.

[16] J. Mierczynski, W. Shen, *Exponential separation and principle Lyapunov exponent/spectrum for random/nonautonomous parabolic equations*, J. Differential Equations, 191(2003), pp 175-205.

[17] B. Moore, "Cochlear Hearing Loss", Whurr Publishers Ltd, London, 2000.

[18] C. Mueller, R. Sowers, *Random Traveling Waves for the KPP equation with Noise*, J. Functional Analysis, 128(1995), pp 439-498.

[19] National Acoustic Labs, *NAL-NL1 software,* 2003.

[20] J. Nolen, J. Xin, *Min-Max Variational Principle and Front Speeds in Random Shear Flows*, Methods and Applications of Analysis, Vol. 11, No. 4, pp 635-644, 2004.

[21] J. Nolen, J. Xin, *A Variational Principle Based Study of KPP Minimal Front Speeds in Random Shears*, Nonlinearity 18 (2005), pp 1655 - 1675.

[22] J. Nolen, J. Xin, *Variational Principle of KPP Front Speeds in Temporally Random Shear Flows*, Comm Math Physics, Vol. 269, No. 2, pp 493-532, 2007.

[23] N. Peters, *Turbulent Combustion*, Cambridge University Press, 2000.

[24] P. Ronney, *Some open issues in premixed turbulent combustion*, in: Modeling in Combustion Science (J. D. Buckmaster and T. Takeno, Eds.), Lecture Notes In Physics, Vol. 449, Springer-Verlag, Berlin, (1995), pp. 3-22.

[25] W. Shen, *Traveling Waves in Diffusive Random Media*, J. Dynamics Diff. Eqs. 16(2004), No. 4, pp 1011-1060.

[26] J. Xin, *Front propagation in heterogeneous media,* SIAM Review, Vol. 42, No. 2, June 2000, pp. 161-230.

[27] J. Xin, *KPP front speeds in random shears and the parabolic Anderson problem*, Methods and Applications of Analysis, Vol. 10, No. 2, (2003), pp. 191-198.

[28] V. Yakhot, *Propagation velocity of premixed turbulent flames*, Comb. Sci. Tech 60, (1988), p. 191.