

A new well-posed nonlinear nonlocal diffusion

Patrick Guidotti*¹

¹ Department of Mathematics, University of California, Irvine, CA 92697, USA

Nonlinear diffusions have successfully been employed to perform a variety of tasks in image processing. In the context of denoising, the Perona-Malik equation is a particularly successful example. In spite of its cherished qualities, it is mathematically ill-posed and has some practical short-comings. In this short paper a new nonlocal nonlinear diffusion is reported upon, which is locally well-posed and shares with Perona-Malik all its qualities even improving on its weaknesses. It can be viewed as a new kind of regularization.

Copyright line will be provided by the publisher

1 The new equation

Since the pioneering contribution of Perona and Malik [5], nonlinear diffusions have enjoyed remarkable success in image processing. In image denoising the task is to rid a noisy image $u_0 = v + n$ of the polluting noise in order to recover the underlying “real” image. Nonlinear diffusions make it possible to adapt the diffusivity to the local properties of the images. Edge preservation is paramount for maintaining image quality. In the Perona-Malik equation

$$\begin{cases} \dot{u} = \nabla \cdot \left(\frac{1}{1+c^2|\nabla u|^2} \nabla u \right), & \text{in } \Omega \text{ for } t > 0, \\ \partial_\nu u = 0, & \text{on } \partial\Omega \text{ for } t > 0, \\ u(0) = u_0, & \text{in } \Omega, \end{cases} \quad (1)$$

this is achieved through a nonlinear diffusion coefficient which decreases in coincidence with sharp gradients. The domain $\Omega \subset \mathbb{R}^2$ can be thought of as the unit square. Such a nonlinear diffusion is ill-posed [4] as it is an instance of a so-called forward-backward diffusion as can be observed by inspecting its linearization. To overcome this difficulty, a variety of regularizations and fixes have been proposed over the years [1, 6]. Here a different paradigm is described which rests on weakening the effect of the nonlinearity through the use of fractional derivatives. Similarly to standard regularizations it leads to a locally well-posed set of equations but, unlike other methods, it preserves the important features of the Perona-Malik model and improves of its shortcomings, such as stair-casing and poor performance in flat regions. The new equation reads

$$\begin{cases} \dot{u} = \frac{1}{1+c^2|(-\Delta)^{1-\varepsilon}u|^2} \Delta u, & \text{in } \Omega \text{ for } t > 0, \\ u \text{ 1-periodic} & \text{for } t > 0, \\ u(0) = u_0, & \text{in } \Omega, \end{cases} \quad (2)$$

where $\varepsilon \in (0, 1)$. At first sight, the close relation between (1) and (2) might not be apparent. In a one dimensional setting, however, the simple change of depend variable

$$v(x) = \int_0^x u(\xi) d\xi$$

reveals their intimate connection when $\varepsilon = 0$. See [2].

2 Analytical Results

For positive ε equation (2) turns out to be globally well-posed.

Theorem 2.1 *Let $u_0 \in C^{2+\alpha}(\Omega)$ for some $\alpha \in (0, 1)$. Then there exist $\beta \in (0, 1)$ and unique local smooth solution*

$$u \in C^{1+\beta}([0, \infty), C^\alpha(\Omega)) \cap C^\beta([0, \infty), C^{2+\alpha}(\Omega))$$

of (2).

A proof of this theorem can be found in [2]. A particularly convenient feature of (2) is that piecewise affine functions are meta-stable stationary solutions. They don't determine the long time behavior of the solution which is controlled by diffusion, but their presence is felt by all solutions. In fact the typical orbit transits in the vicinity of such steady-states at all times on its way to a trivial equilibrium. Standard regularizations of Perona-Malik replace u in the nonlinear diffusivity with a regularized version of it. In (2) the regularization is so mild that it only reduces the singularity produced when applied to continuous piecewise data but does not completely smooth it out.

* Corresponding author: e-mail: gpatrick@math.uci.edu, Phone: +001 949 824 1266, Fax: +001 949 824 7993

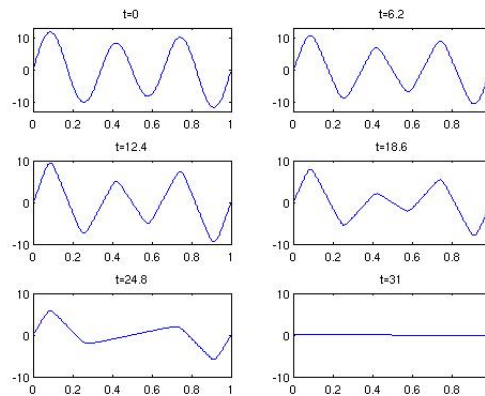


Fig. 1 Typical evolution of a smooth initial datum along a “path of continuous almost piecewise affine functions” according to (2).

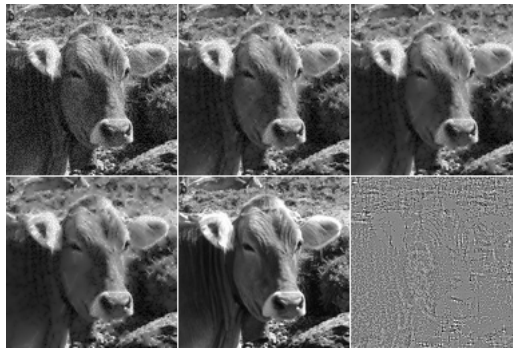


Fig. 2 The denoising effect obtained with (2) with $\varepsilon = 0.4$ and $c = 1$. Depicted are the initial condition, that is, the image of a cow corrupted with about 15% salt and pepper noise, three stages of the denoising progress ($t = 0.15, 0.3, 1.45$), the original image and the difference between the noisy image and its denoised version.

3 Numerical Experiments

In this section numerical experiments are performed that illustrate the described remarkable features of (2). Figure 1 depicts the typical evolution of a smooth initial datum and clearly shows the tendency of the solution to straighten out in between major edges, which are preserved over a longer time. Thus the nonlinearity effectively introduced two time-scales of evolution differentiating between areas of “small” and “large” gradients. Figure 2 illustrates the benefits of (2) when applied to an initial image polluted with about 15% salt and pepper noise. It preserves the qualities of Perona-Malik while alleviating its known short-comings such as “stair-casing” and poor performance in flat regions. All numerical calculations are performed via a pseudo-spectral method which also makes the computation of the fractional derivative term particularly straight-forward (see [3]).

Acknowledgements The author thankfully acknowledges the support of the Swiss Mathematical Society through a travel grant and the National Science Foundation under award DMS-0712875.

References

- [1] A. Buades, B. Coll, and J.M. Morel. A Review of Image Denoising Algorithms, with a New One. *Multiscale Modeling and Simulations*, 4(2):490–530, 2005.
- [2] P. Guidotti. A New Globally Well-posed Nonlinear Nonlocal Diffusion for Noise Reduction. *Submitted*.
- [3] P. Guidotti and J. Lambers. Two New Nonlinear Nonlocal Diffusions for Noise Reduction. *Submitted*.
- [4] S. Kichenassamy. The Perona-Malik paradox. *SIAM J. Appl. Math.*, 57(5):1328–1342, 1997.
- [5] P. Perona and J. Malik. Scale-space and edge detection using anisotropic diffusion. *IEEE Transactions Pattern Anal. Machine Intelligence*, 12:161–192, 1990.
- [6] Joachim Weickert. *Anisotropic Diffusion in Image Processing*. Ph.D. Thesis. Universität Kaiserslautern, Kaiserslautern, 1996.