Singular Elliptic and Parabolic Problems and a Class of Free Boundary Problems

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In this short contribution we report on some recently made progress in the analytical treatment of a class of Free Boundary Problems (FBPs) characterized by the initial onset of a phase. The analysis is motivated by a model problem arising in the description of case II diffusion in polymers. The main local well-posedness and regularity theorem is obtained by means of fundamental results concerning the underlying singular elliptic and parabolic boundary value problems.

1 The Model Problem

A simple model describing case II diffusion in dimensionless variables is given by

$$\begin{cases} \dot{u} - \triangle u = 0\,, & \text{in } \Omega_t \text{ for } t > 0\,, \\ u = g\,, & \text{on } \Gamma_0 \text{ for } t > 0\,, \\ -\partial_\nu u = V\,, & \text{on } \Gamma_t \text{ for } t > 0\,, \\ V = (1 + \delta H)u\,, & \text{on } \Gamma_t \text{ for } t > 0\,, \\ \Gamma\big|_{t=0} = \Gamma_0 \end{cases} \tag{1}$$

where the unknowns u and Γ_t represent the concentration of the diffusing quantity and Γ_t the sharp interface developing in the polymer beyond which this concentration is negligible. V is the front velocity in normal direction ν and H the front's mean curvature. The domain $\Omega_t \subset \mathbb{R}^n \ni (x, y)$ is a strip-like domain between a bottom reservoir $\Gamma_0 = \mathbb{R}^{n-1} \times \{0\}$ and the sharp interface $\Gamma_t = \big\{\big(x, s(t, x)\big) \in \mathbb{R}^n \,\big|\, x \in \mathbb{R}^{n-1}\big\}$. Introducing new variables by rescaling the vertical variable as to map the moving domain to the fixed strip $S = \mathbb{R}^{n-1} \times [0, 1]$, a singularity is introduced in view of the initial condition requiring that $s(0, \cdot) \equiv 0$. The system obtained by dropping the time derivative in the first equation will be referred to as the quasi-stationary approximation of (1).

2 Singular Initial Boundary Value Problems

To leading order in terms of derivatives and singularity intensity the linearization of the diffusion equation for the quantity u is typified by

$$\dot{u} - \triangle_x u - \frac{1}{s^2(t, x)} \partial_{yy} u = f \tag{2}$$

where the singularity affects the diffusivity in a manifestly anisotropic way. Results for such parabolic and elliptic (for the quasi-stationary approximation) problems need to be derived. The diffusive boundary value problem comprising the first three equations in (1) is complemented by either a Hamilton-Jacobi equation (when $\delta = 0$) or a nonlinear parabolic equation (when $\delta > 0$) given by the last two equations. The following result proved in [8] about singular abstract Cauchy problems is essential to the understanding of (2). The space of generators of exponentially decaying analytic semi-groups on a Banach space E_0 is denoted by $\mathcal{H}^-(E_0)$ and the spaces of singular Hölder continuous functions C^β_γ is defined through

$$\mathrm{C}^\beta_\beta = \mathrm{C}^\beta_\beta\big((0, T, E_0\big) := \big\{f \in \mathrm{B}\big((0, T, E_0\big) \,\big|\, [t \mapsto t^\beta f(t)] \in \mathrm{C}^\beta\big((0, T, E_0\big)\big\}$$

where $\mathrm{C}^\beta\big((0, T, E_0\big)$ is the standard space of Hölder continuous functions and

$$\mathrm{C}^\beta_0\big((0, T, E_0\big) = \big\{f \in \mathrm{C}^\beta\big((0, T, E_0\big) \,\big|\, f(0) = 0\big\}$$

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Theorem 2.1 *Assume that A satisfies the following assumptions*

$$\text{(i) } A(t) \in \mathcal{H}^-(E_0,\omega)\,,\; t>0\,, \tag{3}$$

$$\text{(ii) } \|\big[A(t)-A(s)\big]A^{-1}(\tau)\|_{\mathcal{L}(E_0)} \le c\frac{t-s}{t} \text{ and}$$
$$\|\big[A(t)-A(s)\big](-A)^{-\rho}(\tau)\|_{\mathcal{L}(E_0)} \le c(t-s)\,, \tag{4}$$

$$\text{(iii) } \lim_{t\to 0} A^{-1}(t) = 0\,, \tag{5}$$

for some $\rho \in (1,2)$ and $0<\tau \le s\le t\le T$. Let $f \in \mathrm{C}^\beta_\gamma\big((0,T],E_0\big)$ for some $\beta\in(0,1)$ and $\gamma = 0,\beta$. Then $\dot u - A(t)u = f$ has a unique solution $u \in \mathrm{C}^\beta_\gamma\big((0,T],E_0\big)$ satisfying

$$\dot u, Au \in \mathrm{C}^\beta_\gamma \text{ and } \|\dot u\|_{\beta,\gamma} + \|Au\|_{\beta,\gamma} \le c\|f\|_{\beta,\gamma}$$

where $\|\cdot\|_{\beta,0} = \|\cdot\|_\beta$.

Corresponding results for singular boundary value problems need to be derived as well. They are obtained in [6, 7]. The main ingredients are symbol analysis in the constant coefficient case, operator-valued Fourier multipliers and localization arguments.

3 Results

Well-posedness of problems like (1) in a higher dimensional setting have long been open. In the one-dimensional setting they had been solved for some time [1, 2, 3, 5, 4]. Based on the regularity theory briefly referred to above, first higher dimensional local existence results for FBPs like (1) were obtained like the following one from [7].

Theorem 3.1 *For any given $g \in \mathrm{BUC}^{4+\alpha}$ such that $g(x) \ge g_0 > 0$, $x\in\mathbb{R}^{n-1}$, there exists a unique local solution (u,s) of the free boundary problem (1) with $\delta = 0$ such that*

$$u, \dot u, \partial_x^\alpha \frac{1}{t^k}\partial_y^k u \in \mathrm{C}^\beta_\beta\Big((0,T), \mathrm{BUC}^{1+\alpha}\big(\mathbb{R}^{n-1}, \mathrm{C}([0,1])\big)\Big) \text{ for } 0 \le |\alpha| + k \le 2\,,$$
$$\gamma_1 u \in \mathrm{C}^\beta\Big((0,T), \mathrm{BUC}^{3+\alpha}\big(\mathbb{R}^{n-1}\big)\Big)$$
$$\text{and } s \in \mathrm{C}^{1+\beta}\Big((0,T), \mathrm{BUC}^{2+\alpha}\big(\mathbb{R}^{n-1}\big)\Big) \cap \mathrm{C}^\beta\Big((0,T), \mathrm{BUC}^{3+\alpha}\big(\mathbb{R}^{n-1}\big)\Big)\,.$$

where the regularity is measured in the new rescaled variables.

The necessary details and definitions are found in [7]. The quasi-stationary approximation had already been considered in [6] in a two-dimensional setting for all $\delta \ge 0$. Recently a nonlinear stability result for one-dimensional (flat) solutions with respect to two-dimensional perturbations has been proved [9] in a periodic context for the quasi-stationary approximation and for $\delta > 0$. The analysis combines pseudo-differential techniques and a principle of linearized stability for PDEs [10] via a Boundary Integral Formulation.

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