Int. J. Solids Structures, 1968, Vol. 4, pp. 661 to 666. Pergamon Press. Printed in Great Britain

ON THE DISPLACEMENT BOUNDARY VALUE PROBLEM OF SHALLOW SPHERICAL SHELLS*

F. Y. M. WAN

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts

Abstract—By reformulating the displacement boundary conditions in terms of strain and curvature change measures, the displacement boundary value problem of shallow spherical shells becomes the static-geometric analogue of the corresponding stress boundary value problem. Without another set of independent calculations, the exact solution of the former and its asymptotic behavior are obtained directly from known results for the latter by applications of the rules of static-geometric analogy. This new formulation also offers a new perspective to a previous asymptotic solution of the displacement problem.

1. INTRODUCTION

THE asymptotic behavior of the exact solution of the linear elasto-static problem of a complete shallow spherical cap subject to self-equilibrating edge loads was discussed recently in [1]. For the stress boundary value problem, this asymptotic behavior was studied in some detail to delineate the dependence of the interior and edge zone stress state on the applied loads and to determine the applicability of the earlier direct asymptotic solution of the same problem [2, 3]. A less complete analysis of the displacement boundary value problem was also included. The present note reformulates the displacement boundary value problem to explore a duality between the stress and displacement boundary value problem. By formulating the displacement boundary conditions in terms of strain and curvature change measures, the displacement boundary value problem becomes the static geometric analogue of the stress boundary value problem [4]. This complete duality makes separate analysis of the two problems unnecessary. Without another set of independent calculations, the exact solution of the displacement boundary value problem and its asymptotic behavior can be obtained directly from the results for the stress boundary value problem by applying the rules of static geometric analogy. Moreover, our alternate formulation also offers a new perspective to the asymptotic solution obtained in [5].

The fact that there is a static geometric analogy between the two fundamental problems of shell theory was known to Lur'e [6], and a more general exposition of this idea was given by Naghdi in [7] where other appropriate references can also be found. Our present analogy differs from Lur'e's in that the latter requires a reformulation of the stress boundary conditions in terms of resultant forces and moments. Such an analogy does not enable us to make use of the existing results for the stress boundary value problem. An analogy very similar to that of the present work was discussed earlier by Chernykh [8]. However, the starting point of Chernykh's work was the complex form of shell equations of Novozhilov [9]. The derivation of these equations for "complex forces" requires a certain approximation in the original shell equations in the case of a non-vanishing Poisson's ratio. The present analogy does not need this approximation even for non-shallow shells (see [10]).

* Preparation of this article was supported by the Office of Naval Research of the United States Navy.

2. SOLUTION BY STATIC GEOMETRIC ANALOGY

As in [1], we consider solutions of Marguerre's shallow shell equations which lead to finite stresses and displacements at the apex of the shell in the form

$$
\begin{aligned}
w &= \{a^2 A_n \rho^n - [C_n bei_n(\lambda\rho) - D_n ber_n(\lambda\rho)]\} \cos n\theta \\
F &= -\left\{\frac{a^2 B_n}{A(1+\nu_s)}\rho^n + \sqrt{\left(\frac{D}{A}\right)}\,[C_n ber_n(\lambda\rho) + D_n bei_n(\lambda\rho)]\right\} \cos n\theta
\end{aligned}
\tag{1}
$$

with

$$
\rho = \frac{r}{a}, \qquad \lambda = \frac{a}{\sqrt[4]{(\mathrm{DAR}^2)}} \tag{2}
$$

where D and $1/A$ are the bending and stretching stiffness of the shell, R is the radius of the spherical middle surface and where A_n, B_n, C_n, and D_n are constant of integration to be determined by the boundary conditions at the edge $r = a$.* Terms associated with the constants A_n and B_n are the inextensional bending and membrane contributions respectively, where terms associated with C_n and D_n are edge effect contributions.

From (1) and (2), we have the following expressions for the strain and curvature change measures:

$$
\begin{aligned}
\varepsilon_\theta &= A[F_{,rr} - \nu_s(r^{-1}F_{,r} + r^{-2}F_{,\theta\theta})] \\
&= \left\{-n(n-1)B_n\rho^{n-2} + \frac{1}{R}[C_n \hat{g}_{rc}(\lambda\rho) + D_n \hat{g}_{rd}(\lambda\rho)]\right\} \cos n\theta \\
\varepsilon_{r\theta} &= -A(1+\nu_s)(r^{-1}F_{,\theta})_{,r} \\
&= -\left\{n(n-1)B_n\rho^{n-2} + \frac{1}{R}[C_n \hat{g}_{sc}(\lambda\rho) + D_n \hat{g}_{sd}(\lambda\rho)]\right\} \sin n\theta \\
\kappa_r &= -w_{,rr} = \left\{-n(n-1)A_n\rho^{n-2} + \frac{\lambda^2}{a^2}[C_n f_{\theta c}(\lambda\rho) + D_n f_{\theta d}(\lambda\rho)]\right\} \cos n\theta \\
\kappa_\theta &= -(r^{-1}w_{,r} + r^{-2}w_{,\theta\theta}) \\
&= \left\{n(n-1)A_n\rho^{n-2} + \frac{\lambda^2}{a^2}[C_n f_{rc}(\lambda\rho) + D_n f_{rd}(\lambda\rho)]\right\} \cos n\theta \\
\kappa_{r\theta} &= -(r^{-1}w_{,\theta})_{,r} = \left\{n(n-1)A_n\rho^{n-2} + \frac{\lambda^2}{a^2}[C_n f_{sc}(\lambda\rho) + D_n f_{sd}(\lambda\rho)]\right\} \sin n\theta \\
\beta_\theta &= r^{-1}[(r\varepsilon_\theta)_{,r} - \varepsilon_{r\theta,\theta} - \varepsilon_r] = A(\nabla^2 F)_{,r} \\
&= \frac{\lambda}{aR}[C_n bei'_n(\lambda\rho) - D_n ber'_n(\lambda\rho)] \cos n\theta \\
\delta_\theta &= \beta_\theta - r^{-1}\varepsilon_{r\theta,\theta} \\
&= \frac{1}{a}\left\{n^2(n-1)B_n\rho^{n-3} + \frac{\lambda}{R}[C_n \hat{g}_{nc}(\lambda\rho) + D_n \hat{g}_{nd}(\lambda\rho)]\right\} \cos n\theta
\end{aligned}
\tag{3}
$$

* See [1] for nomenclatures not defined herein. Note also the change of notation in regard to the rotations ϕ_r and ϕ_θ and in regard to the constants of integration.

where the functions f's are as defined in [1] while the $\hat{g}$'s are the g's of [1] with v_b replaced by $-v_s$. With the additional analogy associating $(D, -A)$, these are the static geometric analogues of the stress measures given by equations (2.5) of [1]. Within the framework of Marguerre's theory, β_θ can be interpreted as the normal component of the curvature change vector associated with a constant θ edge while δ_θ is the static geometric analogue of the Kirchhoff effective transverse resultant $R_r = Q_r + r^{-1}M_{r\theta,\theta}$.

The shell is subject to edge deformation at $\rho = 1$ so that

$$(u, w, \phi_r) = (u_n, w_n, \phi_n)\cos n\theta, \qquad v = v_n \sin n\theta \tag{4}$$

where for shallow shell theory $\phi_r = -w_{,r}$ and where w_n, u_n, v_n and ϕ_n are prescribed constants and $n \geq 2$. While the conditions (4) were used in [1] for the determination of the constants of integration, we will replace them here by the equivalent conditions

$$\begin{aligned} \kappa_\theta &= \kappa_n \cos n\theta, & \kappa_{r\theta} &= \tau_n \sin n\theta \\ \varepsilon_\theta &= \frac{v_{,\theta}+u}{r} - \frac{w}{R} = \varepsilon_n \cos n\theta, & \delta_\theta &= \frac{v_{,\theta}-u_{,\theta\theta}}{r^2} - \frac{w_{,r}}{R} = \delta_n \cos n\theta \end{aligned} \tag{5}$$

where

$$\begin{aligned} \kappa_n &= a^{-2}(a\phi_n + n^2 w_n), & \tau_n &= -na^{-2}(a\phi_n + w_n), \\ \varepsilon_n &= a^{-1}(nv_n + u_n - \alpha w_n), & \delta_n &= a^{-2}(\alpha a\phi_n + nv_n + n^2 u_n) \end{aligned}$$

and where $\alpha = a/R$, in order to explore the static geometric analogy.

With the relevant strain and curvature change measures given by (3), the boundary conditions (5) become

$$\begin{aligned} n(n-1)A_n + \frac{\lambda^2}{a^2}[C_n f_{rc}(\lambda) + D_n f_{rd}(\lambda)] &= \kappa_n \\ n(n-1)A_n + \frac{\lambda^2}{a^2}[C_n f_{sc}(\lambda) + D_n f_{sd}(\lambda)] &= \tau_n \\ n^2(n-1)B_n + \frac{\lambda}{R}[C_n \hat{g}_{nc}(\lambda) + D_n \hat{g}_{nd}(\lambda)] &= a\delta_n \\ -n(n-1)B_n + \frac{1}{R}[C_n \hat{g}_{rc}(\lambda) + D_n \hat{g}_{rd}(\lambda)] &= \varepsilon_n. \end{aligned} \tag{6}$$

Except for a slight change in notation, equations (6) are formally the same as equations (3.2) of [1] with $-N_n$, S_n, R_n, M_n replaced by κ_n, τ_n, δ_n, ε_n, and with the g's replaced by $\hat{g}$'s. We can therefore write down the solution of (6) without additional calculations

$$\begin{aligned} A_n &= \frac{1}{2n(n-1)}\left[\tau_n \hat{X}_1 + \kappa_n \hat{X}_2 + \frac{n-1}{\alpha a}(a\delta_n + n\varepsilon_n)\hat{X}_3\right] \\ B_n &= \frac{\alpha a}{2n^2(n^2-1)}\left[(\tau_n - \kappa_n)\hat{X}_4 + \frac{n+1}{\alpha a}(a\delta_n \hat{X}_1 - n\varepsilon_n \hat{X}_2)\right] \\ C_n &= \frac{a^2}{q_n\hat{\Delta}_1}\left\{(\tau_n - \kappa_n)\left[\hat{g}_{nd}(\lambda) + \frac{n}{\lambda}\hat{g}_{rd}(\lambda)\right] - \frac{\lambda}{\alpha a}(a\delta_n + n\varepsilon_n)[f_{rd}(\lambda) - f_{sd}(\lambda)]\right\} \\ D_n &= \frac{a^2}{q_n\hat{\Delta}_1}\left\{(\tau_n - \kappa_n)\left[\hat{g}_{nc}(\lambda) + \frac{n}{\lambda}\hat{g}_{rc}(\lambda)\right] - \frac{\lambda}{\alpha a}(a\delta_n + n\varepsilon_n)[f_{rc}(\lambda) - f_{sc}(\lambda)]\right\} \end{aligned} \tag{7}$$

where the $\hat{X}_i$'s and the $\hat{\Delta}_1$ are the X_i's and the Δ_1 of [1] with ν_b replaced by $-\nu_s$ and where q_n is as defined in [1].

3. INTERIOR MEMBRANE AND BENDING STRESSES

To examine the direct and bending stresses in the interior of the shell, we confine ourselves to an isotropic and homogeneous medium for which $\nu_b = \nu_s = \nu$, $A = 1/Eh$ and $D = Eh^3/12(1-\nu^2)$, and consider as in [1] two representative quantities σ^i_{Dn} and σ^i_{Bn} defined by

$$\left.\frac{N^i_\theta}{h}\right|_{\rho=1} = \sigma^i_{Dn}\cos n\theta, \qquad \left.\frac{6M^i_r}{h^2}\right|_{\rho=1} = \sigma^i_{Bn}\cos n\theta. \tag{8}$$

It follows from (7) that (with $\lambda = \mu\sqrt{2}$)

$$\begin{aligned}\frac{\sigma^i_{Bn}}{\sigma^i_{Dn}} &= \frac{6(1-\nu^2)DA}{h}\frac{A_n}{B_n}\\ &= \frac{n(n+1)\sqrt{[3(1-\nu^2)]}}{2\mu^2}\left[\frac{\tau_n\hat{X}_1+\kappa_n\hat{X}_2+[(n-1)/\alpha a](a\delta_n+n\varepsilon_n)\hat{X}_3}{(\tau_n-\kappa_n)\hat{X}_4+[(n+1)/\alpha a](a\delta_n\hat{X}_1-n\varepsilon_n\hat{X}_2)}\right]\end{aligned} \tag{9}$$

which gives a measure of the relative magnitude of the interior direct and bending stresses. Equation (9) is formally the same as the corresponding equation for the stress boundary value problem with the role of σ^i_{Bn} and σ^i_{Dn} interchanged and with the prescribed edge stress resultants and couple replaced by the prescribed edge strain couples and resultant according to the rules of the static geometric analogy. Observing these changes, a discussion of the interior stress state for the displacement boundary value problem is formally the same as that of the stress boundary value problem in [1]. We confine ourselves only to the following observations.

From equation (3.2) of [1], we have the following asymptotic expansions for the $\hat{X}_i$'s for $\mu \gg 1$:

$$\begin{aligned}&\hat{X}_1 \sim 1+\frac{(n-1)(1+\nu)}{2\mu}+O\left(\frac{1}{\mu^2}\right), &&\hat{X}_2 \sim 1-\frac{(n-1)(1+\nu)}{2\mu}+O\left(\frac{1}{\mu^2}\right)\\ &\hat{X}_3 \sim 1+\frac{(n+1)(1+\nu)}{2\mu}+O\left(\frac{1}{\mu^2}\right), &&\hat{X}_4 \sim 1+\frac{(n-1)(1+\nu)}{2\mu}+O\left(\frac{1}{\mu^2}\right).\end{aligned} \tag{10}$$

With these, there follows from (7)

$$\begin{aligned}A_n &= \frac{1}{2n(n-1)}\left\{\left[1+\frac{(n-1)(1+\nu)}{2\mu}+O\left(\frac{1}{\mu^2}\right)\right]\tau_n+\left[1-\frac{(n-1)(1+\nu)}{2\mu}+O\left(\frac{1}{\mu^2}\right)\right]\kappa_n\right.\\ &\qquad\left.+\frac{n-1}{\alpha a}\left[1+\frac{(n+1)(1+\nu)}{2\mu}+O\left(\frac{1}{\mu^2}\right)\right](a\delta_n+n\varepsilon_n)\right\}\\ &= \frac{1}{2n(n-1)}\left\{\left[1+O\left(\frac{1}{\mu^2}\right)\right](\tau_n+\kappa_n)+\frac{(n-1)(1+\nu)}{2\mu}\left[1+O\left(\frac{1}{\mu}\right)\right](\tau_n-\kappa_n)\right.\\ &\qquad\left.+\frac{(n-1)}{\alpha a}\left[1+O\left(\frac{1}{\mu}\right)\right](\alpha\delta_n+n\varepsilon_n)\right\}\end{aligned} \tag{11}$$

$$B_n = \frac{h\mu^2}{2n^2(n^2-1)\sqrt{[3(1-\nu^2)]}}\left\{\left[1+\frac{(n-1)(1+\nu)}{2\mu}+O\left(\frac{1}{\mu^2}\right)\right](\tau_n-\kappa_n)\right.$$
$$\left.+\frac{n+1}{\alpha a}\left[1+\frac{(n-1)(1+\nu)}{2\mu}+O\left(\frac{1}{\mu^2}\right)\right](a\delta_n)-\frac{n+1}{\alpha a}\left[1-\frac{(n-1)(1+\nu)}{2\mu}+O\left(\frac{1}{\mu^2}\right)\right](n\varepsilon_n)\right\}$$
$$= \frac{h\mu^2}{2n^2(n^2-1)\sqrt{[3(1-\nu^2)]}}\left\{\left[1+O\left(\frac{1}{\mu}\right)\right](\tau_n-\kappa_n)+\frac{n+1}{\alpha a}\left[1+O\left(\frac{1}{\mu^2}\right)\right](a\delta_n-n\varepsilon_n)\right.$$
$$\left.+\frac{(n^2-1)(1+\nu)}{2\mu\alpha a}\left[1+O\left(\frac{1}{\mu}\right)\right](a\delta_n+n\varepsilon_n)\right\}.$$

If (11) is an accurate approximation of A_n and B_n and if the prescribed edge displacements are such that

$$\frac{1}{\mu}\left|(\tau_n-\kappa_n)+\frac{n+1}{\alpha a}(a\delta_n+n\varepsilon_n)\right| \ll \left|(\tau_n+\kappa_n)+\frac{n-1}{\alpha a}(a\delta_n+n\varepsilon_n)\right|$$

and (12)

$$\frac{1}{\mu}\left|(\tau_n-\kappa_n)+\frac{n+1}{\alpha a}(a\delta_n+n\varepsilon_n)\right| \ll \left|(\tau_n-\kappa_n)+\frac{n+1}{\alpha a}(a\delta_n-n\varepsilon_n)\right|$$

we have the following valid first approximation for A_n and B_n:

$$A_n = \frac{1}{2n(n-1)}\left[(\tau_n+\kappa_n)+\frac{n-1}{\alpha a}(a\delta_n+n\varepsilon_n)\right] = \frac{n+1}{2\alpha a^2}(u_n+v_n)$$
$$B_n = \frac{\alpha a}{2n^2(n^2-1)}\left[(\tau_n-\kappa_n)+\frac{n+1}{\alpha a}(a\delta_n-n\varepsilon_n)\right] = \frac{u_n-v_n}{2na}. \tag{13}$$

It is remarkable that the quantities w_n and ϕ_n which contribute to each edge strain measure individually do not appear in the leading terms of the interior solution. Moreover, the expressions for A_n and B_n given by (13) are exactly those obtained in [5] by a direct asymptotic analysis of the same problem. Our analysis shows that the results of [5] are in fact equivalent to the leading term of the asymptotic expansion of the interior solution in powers of $1/\mu$. Whether such a leading term theory provides an accurate first approximation of the exact interior solution depends of course on the satisfaction of (12) and on that (11) is an accurate approximation of the exact solution.

Writing (12) in terms of the edge displacement quantities, we have

$$\mu^{-1}|(n+1)(u_n+v_n)-2\alpha w_n| \ll (n-1)|u_n+v_n|$$
$$\mu^{-1}|(n+1)(u_n+v_n)-2\alpha w_n| \ll (n-1)|u_n-v_n|. \tag{14}$$

The first condition of (14) can be simplified to $|\alpha w_n/\mu| \ll (n-1)|u_n+v_n|$. Note that ϕ_n does not appear in (14). This is in agreement with the result of [1] which shows that the contribution of the edge rotation is of the form $\alpha a\phi_n/(2\mu)^2$. Thus, even if (14) (or (12)) is satisfied, (13) need not be an accurate first approximation of these constants. However, upon appropriate translation of the results for the stress boundary value problem, we have that if (13) is not an accurate first approximation of the interior stress state, then it is the edge zone stress rather than the interior stress state which is associated with the dominant stress of