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Inertial Manifolds and Gevrey Regularity for the Moore-Greitzer Model of an Axial-Flow Compressor

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Summary. In this paper, we study the regularity and long-time behavior of the solutions to the Moore-Greitzer model of an axial-flow compressor. In particular, we prove that this dissipative system of evolution equations possesses a global invariant inertial manifold, and therefore its underlying long-time dynamics reduces to that of an ordinary differential system. Furthermore, we show that the solutions of this model belong to a Gevrey class of regularity (real analytic in the spatial variables). As a result, one can show the exponentially fast convergence of the Galerkin approximation method to the exact solution, an evidence of the reliability of the Galerkin method as a computational scheme in this case. The rigorous results presented here justify the readily available low-dimensional numerical experiments and control designs for stabilizing certain states and traveling wave solutions for this model.

1. Introduction

The study of air flow through compressors, which is important to turbo-machine engines, has drawn a lot of attention in recent years. It has been observed that there are certain parameter values at which jet engines operate optimally. However, when the mass flow through the compressor reaches some critical value, the steady air flow is no longer stable and as a result the engine has to be shut down and restarted again. Due to these instabilities, engines are currently designed to be operated away from their optimal parameter values. Instability problems of compressors have been a major concern in the design of aircraft engines, and it is a challenging task to stabilize the air flow while improving the performance of jet engines.

In [34] Moore and Greitzer introduced a successful PDE model for an axial-flow compressor. To better understand the true nature of the flow in an axial-flow compressor, others have also suggested modifications to this model. For instance, viscous dissipation of energy in the compressor is accounted in an ad hoc fashion by Adomaitis and Abed [1]. Recently, Mezić [33] argued that this ad hoc dissipation can be derived analytically from the eddy viscosity term in classical turbulence modeling. Several intensive computational studies of these modified models have been performed (see, e.g., [1], [2], [3], [31], [32], and [35]). In particular, McCaughan [32] analyzes the nature of flow from a bifurcation analysis point of view. Furthermore, in an attempt to stabilize certain states, Mansoux et al. [31] introduced a model suitable for control analysis and design. Noticeably, from [31], [32], and references therein, it is observed that the computational approach using a low-dimensional Galerkin approximation showed reasonably good agreement with experimental data. However, it has not yet been justified analytically why a very low-dimensional Galerkin approximation captures the right dynamical behavior of this infinite-dimensional dynamical system. In this article, we attempt to answer this question and give a rigorous analytical explanation of these numerical experiments. Specifically, we will prove that the exact solutions to these equations belong to a certain Gevrey class of regularity (the real analytic functions with respect to the spatial variable with uniform lower bound for their radii of analyticity). As an immediate consequence of this regularity result, one should be able to prove the exponential rate of convergence of the Galerkin procedure (see, e.g., [10], [11], [16], and [21]).

The precise definition of Gevrey regularity can be found in [19] (see p. 73 and Section 2.3 of [19]). As we have mentioned above, here, we are considering a special class of Gevrey regularity, namely, the real analytic functions, which is equivalent to Gevrey class 1 in [19]. A more detailed mathematical explanation on the relationship between this special Gevrey class and the general concept of the Gevrey class can be found in [26]. There have been many studies regarding this special Gevrey regularity of solutions to dissipative nonlinear PDEs (see, e.g., [10], [11], [12], [15], [18], [23], [26], [36], [38], [39], and references therein). Foias and Temam [15] show the Gevrey regularity of solutions of the Navier-Stokes equation. Inspired by [15], the authors of [10] and [11] applied this technique to the complex Ginzburg-Landau equation and proved rigorously the exponential convergence rate of the Galerkin procedure. Promislow [39] established similar regularity results for nonlinear parabolic equations with a polynomial nonlinearity using the techniques of [15]. In [12] the authors succeeded in further extending the applicability of the Foias and Temam [15] technique to establish the Gevrey regularity for general nonlinear analytic parabolic PDEs. All the above mentioned work has been done for parabolic-type equations. However, other nonparabolic, yet dissipative, equations enjoy the same type of regularity asymptotically in time. Such examples include the damped driven nonlinear Schrödinger equation [36] and the Bénard convection in porous medium [37] (see also [28]).

Before attempting to prove the regularity of solutions, one has to establish the existence and uniqueness of these solutions. The question of well-posedness (existence, uniqueness, and continuous dependence on initial data) has already been established by Birnir and Hauksson [3]. Here, however, we use different methods than those employed in [3]. The techniques adopted in this paper enable us to improve the regularity of the solutions to the Moore-Greitzer model (we refer the readers to [5] for more details). For the proof of a Gevrey class of regularity, we use tools which are inspired by the work of Foias and Temam [15] (see also [12]). These approaches simply rely on the energy estimate and the parabolic smoothing effect of the underlying equation. However, it is worth mentioning that similar results concerning the Burgers and Navier-Stokes equations have been established independently by Kreiss [23] and Henshaw et al. [18], but the techniques used there are very different from those in [15] and [12].

In an attempt to interpret the low-dimensional dynamical behavior of the Moore-Greitzer model, which was observed experimentally, Birnir and Hauksson [3] recently showed that this model possesses a finite-dimensional global attractor. Here we make further progress in this direction and establish the existence of a C^1 finite-dimensional globally invariant inertial manifold, which contains the global attractor. Thus the long-time dynamics of the Moore-Greitzer model can be described by that of a finite-dimensional ordinary differential system.

An inertial manifold (IM) is a positively (in time) invariant C^1 finite-dimensional manifold, which attracts exponentially all bounded subsets of the phase space. On the IM, the PDE reduces to a finite-dimensional system of ODEs, called an inertial form. This means that all the dynamical properties of the PDE, such as bifurcation and stability, can be described by studying the reduced inertial form. The estimates for the dimension of the global attractor depend inversely on the size of the viscosity; hence they are very large for small viscosity. The dimension of the IM suffers from the same disadvantage. However, it is worth stressing that from the rigorous mathematical point of view, by proving the existence of an IM we are making the reduction from an infinite- to a finite-dimensional system. This is an infinite jump even though the estimates provided here are very large. Let us observe, however, that this is only an attempt at explaining the finite-dimensional behavior of this model, and further work is needed in order to be able to provide more realistic estimates.

As we mentioned before, in order to improve the performance of jet engines with better fuel efficiency, we must design a control device to overcome flow instabilities in the compressor. The design of controllers by considering Liapunov functions is suggested, for example, in [2] and [4] (see also references therein). Here we propose to approach this problem from a different direction by taking advantage of the finite-dimensional asymptotic behavior, i.e., the existence of an inertial manifold, of the Moore-Greitzer model in designing feedback controllers, a subject of ongoing research.

2. The Physical Model

First let us examine some of the elementary features of jet engines and the assumption that Moore and Greitzer imposed on the model. The basic compression system is composed of an inlet duct, compressor, plenum, and throttle. The incoming air from the inlet duct is compressed and discharged to the plenum. The flow is controlled by the throttle. Moore and Greitzer [34] assumed that the compressor is considered to have a high hub-to-tip ratio (smaller than but very close to 1), i.e., roughly speaking, very short blades in comparison to the radius of the hub. Thus the radial variations are negligible, and therefore the compression system can be considered to be two-dimensional and described by the axial and angular coordinates. The pressure in the plenum is assumed to be uniform spatially, but time dependent, and the gas in the plenum to be compressible. Let θ, η, t represent circumferential coordinate, axial coordinate, and time, respectively. In the inlet, irrotational flow is assumed, i.e., the velocity potential $\phi(\eta, \theta, t)$ satisfies Laplace's equation. We decompose the velocity potential $\phi(\eta, \theta, t) = \Phi(t) + \phi'(\eta, \theta, t)$, where $\phi'(\eta, \theta, t)$ is the potential for the disturbance from the mean flow at the entrance duct $\Phi(t) = \frac{1}{2\pi}\int_0^{2\pi} \phi(0, \theta, t)\, d\theta$. We now introduce the Moore-Greitzer model [34]:

$$\phi'_{\eta\eta} + \phi'_{\theta\theta} = 0,$$

on the cylinder $(\eta, \theta) \in (-\infty, 0) \times [0, 2\pi]$ with zero boundary conditions at $\eta \to -\infty$, i.e., $\lim_{\eta\to-\infty} \phi'(\eta, \theta, t) = 0$, and

$$\begin{aligned}
\Psi(t) &= \psi_c(\Phi + \phi'_\eta|_{\eta=0}) - l_c\frac{d\Phi}{dt} - m\phi'_t|_{\eta=0} \\
&\quad - \frac{1}{2a}(2\phi'_{t\eta} + \phi'_{\theta\eta})|_{\eta=0} + \frac{\nu}{2}\phi'_{\eta\theta\theta}|_{\eta=0} \qquad \text{(1a)} \\
l_c\frac{d\Phi}{dt} &= -\Psi(t) + \frac{1}{2\pi}\int_0^{2\pi} \psi_c(\Phi + \phi'_\eta|_{\eta=0})\, d\theta \qquad \text{(1b)} \\
l_c\frac{d\Psi}{dt} &= \frac{1}{4B^2}(\Phi(t) - F_T^{-1}(\Psi)). \qquad \text{(1c)}
\end{aligned}$$

$\Psi(t)$ is the mean pressure over the compressor. ψ_c and F_T^{-1} are the pressure change over the compressor and the throttle, respectively.

Since the disturbance potential vanishes as $\eta \to -\infty$, the Laplace's equation has a solution of the form

$$\phi'(\eta, \theta, t) = \sum_{n\in\mathbb{Z}\backslash 0} \alpha_n(t) e^{|n|\eta - in\theta}.$$

Following [2], we now rewrite the model by using the above solution. First let us define $\varphi(\theta, t) = \phi'_\eta|_{\eta=0} = \sum_{n\in\mathbb{Z}\backslash 0} \alpha_n(t)|n|e^{-in\theta}$. Substituting (1b) into (1a) and making a use of $\varphi(\theta, t)$ instead of $\phi'(\eta, \theta, t)$, the study of the Moore-Greitzer model reduces to investigating the following nonlinear, nonlocal, pseudodifferential system [2]

$$\begin{aligned}
\frac{\partial}{\partial t}\varphi &= K^{-1}\left(\frac{\nu}{2}\frac{\partial^2\varphi}{\partial\theta^2} - \frac{1}{2}\frac{\partial\varphi}{\partial\theta} + a(\psi_c(\Phi + \varphi) - \overline{\psi_c})\right), \qquad \text{(2a)} \\
\frac{d\Phi}{dt} &= \frac{1}{l_c}(\overline{\psi_c} - \Psi), \qquad \text{(2b)} \\
\frac{d\Psi}{dt} &= \frac{1}{4l_cB^2}(\Phi - F_T^{-1}(\Psi)), \qquad \text{(2c)} \\
(\varphi, \Phi, \Psi)|_{t=0} &= (\varphi_0, \Phi_0, \Psi_0), \qquad \text{(2d)}
\end{aligned}$$

in the domain $\Omega = [0, 2\pi]$ subject to periodic boundary conditions. The above is a coupled system of equations with one PDE and 2 ODEs.

Here $F_T^{-1}(\Psi)$ is assumed to be a smooth function, which is defined as $F_T^{-1}(\Psi) = \gamma\Psi|\Psi|^{-\frac{1}{2}}$ outside a small neighborhood of $\Psi = 0$. However, this function can be modified based on the throttle parameter.

For every $\chi \in L^2([0, 2\pi])$ with Fourier expansion $\chi(\theta) = \sum_{n\in\mathbb{Z}} \hat{\chi}_n e^{in\theta}$, the operator K is defined by

$$K\left(\hat{\chi}_0 + \sum_{n\in\mathbb{Z}\backslash\{0\}} \hat{\chi}_n e^{in\theta}\right) = \hat{\chi}_0 + \sum_{n\in\mathbb{Z}\backslash\{0\}} \left(1 + \frac{am}{|n|}\right) \hat{\chi}_n e^{in\theta}.$$

ψ_c is an empirical function. Here we assume that ψ_c is a cubic polynomial in $\Phi + \varphi$,

$$\psi_c(\Phi + \varphi) = -k_0(\Phi + \varphi)^3 + k_1(\Phi + \varphi)^2 + k_2(\Phi + \varphi) + k_3, \tag{3}$$

with $k_0 > 0$, and the average of ψ_c is defined by $\overline{\psi_c}(t) = \frac{1}{2\pi}\int_0^{2\pi} \psi_c(\Phi(t) + \varphi(\theta, t))\, d\theta$. Observe that the average of φ, i.e., $\bar{\varphi}(t) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(\theta, t)\, d\theta$, satisfies $\frac{d\bar{\varphi}}{dt} = 0$. Thus, $\bar{\varphi}(t) = \bar{\varphi}(0)$. Here we assume $\overline{\varphi_0} = 0$, and hence $\bar{\varphi}(t) = 0$ for all $t \geq 0$.

3. Preliminaries and Notations

Throughout this paper, we denote $\mathbb{H} = \{g \in L^2(\Omega)\colon \int_\Omega g(\theta)\ d\theta = 0\}$, and $\mathbb{V} = \mathbb{H} \cap H^1_{per}(\Omega)$. $(\cdot, \cdot)$,$((\cdot, \cdot))$ are the $\mathbb{H}$ and $\mathbb{V}$ inner products respectively, given by

$$(g, h) = \int_\Omega g(\theta)h(\theta)\ d\theta, \quad ((g, h)) = \int_\Omega \frac{\partial g(\theta)}{\partial\theta}\frac{\partial h(\theta)}{\partial\theta}\ d\theta.$$

$|\cdot|$, $\|\cdot\|$ denote the $\mathbb{H}$ and $\mathbb{V}$ norm respectively. We also denote by $\mathbb{V}'$ the dual space of $\mathbb{V}$. Next we introduce the Hilbert spaces $\mathcal{H} = \mathbb{H} \times \mathbb{R} \times \mathbb{R}$, and $\mathcal{V} = \mathbb{V} \times \mathbb{R} \times \mathbb{R}$ with the appropriate inner products induced by the inner products of $\mathbb{H}$, $\mathbb{V}$, and $\mathbb{R}$, which lead to the corresponding norms

$$|(\varphi, \Phi, \Psi)|_{\mathcal{H}} = r_1|\varphi| + r_2|\Phi| + r_3|\Psi|, \quad \text{for every } (\varphi, \Phi, \Psi) \in \mathcal{H}, \tag{4a}$$

$$\|(\varphi, \Phi, \Psi)\|_{\mathcal{V}} = \|\varphi\| + |\Phi| + |\Psi|, \quad \text{for every } (\varphi, \Phi, \Psi) \in \mathcal{V}, \tag{4b}$$

where r_1, r_2, r_3 are some positive constants which will be determined later in Theorem 1. For $\boldsymbol{w} \in \mathcal{H}, \mathcal{V}$ we will denote $|\boldsymbol{w}|_{\mathcal{H}} = |\boldsymbol{w}|$ and $|\boldsymbol{w}|_{\mathcal{V}} = \|\boldsymbol{w}\|$ respectively. We will also denote $\boldsymbol{w} \in \mathcal{H}, \mathcal{V}$ by $(\cdot, \cdot)$ the induced inner product in $\mathcal{H}$. Observe that K, K^{-1} are bounded linear positive operators on $\mathbb{H} \times \mathbb{R}$, $\mathbb{V} \times \mathbb{R}$, and $\mathbb{V}' \times \mathbb{R}$ satisfying

$$c_1^{-1}(|\varphi| + |\Phi|) \leq |K(\varphi + \Phi)|_{L^2} \leq c_1(|\varphi| + |\Phi|), \quad \text{for every } (\varphi, \Phi) \in \mathbb{H} \times \mathbb{R},$$

and

$$c_2^{-1}(\|\varphi\| + |\Phi|) \leq \|K(\varphi + \Phi)\|_{H^1} \leq c_2(\|\varphi\| + |\Phi|), \quad \text{for every } (\varphi, \Phi) \in \mathbb{V} \times \mathbb{R},$$

where c_1 and c_2 are two positive constants.

By letting $\boldsymbol{u} = (\varphi, \Phi, \Psi)$ the system (2) can be written as a functional differential equation

$$\frac{d}{dt}\boldsymbol{u} = \boldsymbol{L}\boldsymbol{u} + \mathbf{f}(\boldsymbol{u}), \tag{5a}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \tag{5b}$$

where

$$\boldsymbol{L}\boldsymbol{u} = \left(K^{-1}\left(-\frac{\nu}{2}A\varphi - \frac{1}{2}\frac{\partial\varphi}{\partial\theta} \right), 0, 0 \right),$$

$$\mathbf{f}(\boldsymbol{u}) = \left(aK^{-1}(\psi_c(\Phi+\varphi) - \overline{\psi_c}), \frac{1}{l_c}(\overline{\psi_c} - \Psi), \frac{1}{4l_cB^2}(\Phi - F_T^{-1}(\Psi)) \right).$$

Here, $A = -\frac{\partial^2}{\partial\theta^2}$ with $D(A) = \mathbb{H} \cap H^2_{per}(\Omega)$, where $H^m_{per}(\Omega)$ denotes the usual L^2 Sobolev spaces subject to periodic boundary conditions. Furthermore, A has a discrete spectrum of eigenvalues $\lambda_j = j^2$, $j = 1, 2, \ldots$, each with multiplicity 2.

The above assumptions on K, ψ_c, and F_T^{-1} guarantee that the nonlinearity $\mathbf{f}(\boldsymbol{u})$ is differentiable with respect to the variable $\boldsymbol{u}$, when $\boldsymbol{u}$ belongs to the appropriate spaces. $\boldsymbol{L}$ has the domain $D(\boldsymbol{L}) = D(A) \times \mathbb{R} \times \mathbb{R}$. Denoting by $\mathcal{V}' = \mathbb{V}' \times \mathbb{R} \times \mathbb{R}$ the dual space of $\mathcal{V}$, and by $\langle\cdot,\cdot\rangle$ the dual action of $\mathcal{V}'$ on elements of $\mathcal{V}$, the operator $\boldsymbol{L}$ can be extended in a unique fashion to $\boldsymbol{L}\colon \mathcal{V} \to \mathcal{V}'$ such that

$$\langle \boldsymbol{L}\boldsymbol{u}, \boldsymbol{v}\rangle = -\frac{\nu}{2}((K^{-1}\varphi_{\boldsymbol{u}}, \varphi_{\boldsymbol{v}})) + \frac{1}{2}\left(K^{-1}\varphi_{\boldsymbol{u}}, \frac{\partial\varphi_{\boldsymbol{v}}}{\partial\theta} \right), \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}, \tag{6}$$

where $\boldsymbol{u} = (\varphi_{\boldsymbol{u}}, \Phi_{\boldsymbol{u}}, \Psi_{\boldsymbol{u}})$ and $\boldsymbol{v} = (\varphi_{\boldsymbol{v}}, \Phi_{\boldsymbol{v}}, \Psi_{\boldsymbol{v}})$. Hence, $\langle \boldsymbol{L}\boldsymbol{u}, \boldsymbol{u}\rangle = -\frac{\nu}{2}||K^{-\frac{1}{2}}\varphi_{\boldsymbol{u}}||^2$, for all $\boldsymbol{u} \in \mathcal{V}$. This extension is necessary when we talk later about weak solutions. Throughout the paper, we denote by $c_0, c_1, \ldots, C_0, C_1, \ldots, M_0, M_1, \ldots$ positive constants independent of the time variable. Now we introduce some definitions.

Definition 1. Let $T > 0$ and $\boldsymbol{u}_0 \in \mathcal{H}$ be given. A weak solution of equation (5) on the interval $[0, T]$ is a function $\boldsymbol{u} \in L^2([0,T];\mathcal{V}) \cap C([0,T];\mathcal{H})$ satisfying $\frac{d\boldsymbol{u}}{dt} \in L^2([0,T];\mathcal{V}')$ and

$$\frac{d}{dt}\langle \boldsymbol{u}, v\rangle = \langle \boldsymbol{L}\boldsymbol{u}, v\rangle + \langle \mathbf{f}(\boldsymbol{u}), v\rangle, \quad \text{in } \mathcal{D}'(0,T),$$

(i.e., in the distribution sense on $(0, T)$), for all $v \in \mathcal{V}$,

and

$$\boldsymbol{u}(0) = \boldsymbol{u}_0.$$

Definition 2. Let $T > 0$ and $\boldsymbol{u}_0 \in \mathcal{V}$ be given. A strong solution of equation (5) on the interval $[0, T]$ is a function $\boldsymbol{u} \in L^2([0,T]; D(A) \times \mathbb{R} \times \mathbb{R}) \cap C([0,T];\mathcal{V})$ satisfying $\frac{d\boldsymbol{u}}{dt} \in L^2([0,T];\mathcal{H})$ and

$$\frac{d}{dt}(\boldsymbol{u}, v) = (\boldsymbol{L}\boldsymbol{u}, v) + (\mathbf{f}(\boldsymbol{u}), v), \quad \text{in } \mathcal{D}'(0,T),$$

(i.e., in the distribution sense on $(0, T)$), for all $v \in \mathcal{V}$,

and

$$\boldsymbol{u}(0) = \boldsymbol{u}_0.$$

Remark 1. According to Lemma 1.1, Chapter 3, [40], Definition 1 implies that

$$\begin{aligned} \boldsymbol{u}' &= \boldsymbol{L}(\boldsymbol{u}) + \mathbf{f}(\boldsymbol{u}) \quad \text{in } \mathcal{V}' \text{ a.e. for } t \in [0,T], \\ \text{i.e., } \boldsymbol{u}(t) &= \boldsymbol{u}(0) + \int_0^t (\boldsymbol{L}\boldsymbol{u}(s) + \mathbf{f}(\boldsymbol{u}(s)))\, ds \quad \text{in } \mathcal{V}' \text{ a.e. for } t \in [0,T]. \end{aligned}$$

Similarly, Definition 2 implies that $\boldsymbol{u}' = \boldsymbol{L}(\boldsymbol{u}) + \mathbf{f}(\boldsymbol{u})$ in $\mathcal{H}$ a.e. for $t \in [0,T]$, i.e., $\boldsymbol{u}(t) = \boldsymbol{u}(0) + \int_0^t (\boldsymbol{L}\boldsymbol{u}(s) + \mathbf{f}(\boldsymbol{u}(s)))\, ds$ in $\mathcal{H}$ a.e. for $t \in [0,T]$.

Remark 2. Since we required $\varphi(t) \in \mathbb{V}$, a.e. for $t \in [0,T]$ in Definition 1 and Definition 2 and since $\mathbb{V} \subset H^1(\Omega) \subset L^\infty(\Omega)$ is an algebra, the nonlinear terms in equation (5) make sense.

4. Existence and Uniqueness of Weak and Strong Solutions

In this section we show the global existence, uniqueness, and well-posedness of weak and strong solutions to system (5). For $\boldsymbol{u}_0 \in \mathcal{H}$ or $\boldsymbol{u}_0 \in \mathcal{V}$ we will denote by $S(t)\boldsymbol{u}_0 = \boldsymbol{u}(t)$ the semigroup of solution operator. In addition, we prove the existence of an absorbing ball in $\mathcal{H}$, $\mathcal{V}$ and $D(A) \times \mathbb{R} \times \mathbb{R}$. In order to achieve that, we shall implement the Galerkin method. This approach is an alternative to the proof given in [3], and it will be used to improve some of the regularity results of [3]. Once we establish the existence and the uniqueness of a solution to the Galerkin approximate system, we need to establish some a priori estimates on the various norms of the Galerkin solution, which are independent of the size of the Galerkin system. Using the appropriate compactness theorem (Aubin's Compactness Theorem, see e.g., [6], [27], or [40]), we can extract adequate subsequences which converge to a limit function, which in turn will be proved to be a weak solution to the equation (5). Then by passing to the limit with these a priori estimates for the Galerkin system, we establish these same estimates for the limit function, i.e., to a solution of (5). And only after demonstrating the uniqueness of weak solutions do we reach these estimates for the unique weak solution. The same arguments are applied to the proof of the strong solution.

Let us denote by P_N the L^2 orthogonal projection from $\mathbb{H}$ onto $\mathbb{H}_N$, the span of $\{e^{ij\theta} : 1 \le |j| \le N\}$, and by $\varphi^N(\theta,t) = \sum_{1\le|j|\le N} \hat{\varphi}_j^N(t) e^{ij\theta}$, subject to the reality condition $\hat{\varphi}_{-j}^N = (\hat{\varphi}_j^N)^*$. The sequence $\boldsymbol{u}^N = (\varphi^N, \Phi^N, \Psi^N)$ will denote an approximating solution to system (5), which is defined as the solution of the Galerkin system

$$\frac{d}{dt}\boldsymbol{u}^N = \boldsymbol{L}\boldsymbol{u}^N + P_N \mathbf{f}(\boldsymbol{u}^N), \tag{7}$$

with the initial value $\boldsymbol{u}^N(0) = P_N \boldsymbol{u}_0 = (P_N\varphi_0, \Phi_0, \Psi_0)$, where

$$\boldsymbol{u}^N = (\varphi^N, \Phi^N, \Psi^N), \qquad \boldsymbol{L}\boldsymbol{u}^N = \left(K^{-1}\left(-\frac{\nu}{2}A\varphi^N - \frac{1}{2}\frac{\partial \varphi^N}{\partial\theta}\right), 0, 0\right),$$

and

$$\begin{aligned} P_N \mathbf{f}(\boldsymbol{u}^N) = {} & P_N(aK^{-1}(\psi_c(\Phi^N + \varphi^N) - \overline{\psi_c}(\Phi^N + \varphi^N)), \\ & \frac{1}{l_c}(\overline{\psi_c}(\Phi^N + \varphi^N) - \Psi^N), \frac{1}{4l_cB^2}(\Phi^N - F_T^{-1}(\Psi^N))). \end{aligned}$$

Equation (7) is equivalent to the ODE system

$$\frac{d}{dt}\hat{\varphi}_j^N = K^{-1}\left(\frac{\nu}{2}(ij)^2\hat{\varphi}_j^N - \frac{1}{2}(ij)\hat{\varphi}_j^N + a(\psi_c(\Phi^N + \varphi^N), e^{ij\theta})\right), \tag{8a}$$

$$\frac{d\Phi^N}{dt} = \frac{1}{l_c}(\overline{\psi_c}(\Phi^N + \varphi^N) - \Psi^N), \tag{8b}$$

$$\frac{d\Psi^N}{dt} = \frac{1}{4l_cB^2}(\Phi^N - F_T^{-1}(\Psi^N)), \tag{8c}$$

$$(\hat{\varphi}_j^N, \Phi^N, \Psi^N)|_{t=0} = (\hat{\varphi}_{0,j}, \Phi_0, \Psi_0), \tag{8d}$$

where $\hat{\varphi}_{0,j} = (\varphi_0, e^{-ij\theta})$, for $1 \leq |j| \leq N$.

Since the nonlinear terms are locally Lipschitz, by the theorem of existence and uniqueness for finite ODE systems we conclude that for each N, finite, the system (8), and hence (7), has a unique solution for an open interval of time, around $t = 0$. However, the length of this interval of existence might depend on N. To show the global existence of the Galerkin system (7) for all $t > 0$, it suffices to show that the solutions remain bounded for all $t > 0$. Instead of showing these detailed estimates for the Galerkin system, we will establish them formally and directly on the full system (5), which can be justified rigorously by developing them first for the Galerkin approximate system and then passing to the limit. To be more specific, the $L^\infty([0, T]; \mathcal{H})$ and $L^2([0, T]; \mathcal{V})$ bounds on the exact solutions are established first for the Galerkin, then we validate them for the exact solutions by passing to the limit.

Theorem 1. *Let* $T > 0$ *be given and* $\boldsymbol{u}_0 \in \mathcal{H}$. *There exists a unique weak solution of the system (5) on the interval* $[0, T]$. *Furthermore, there exists a constant* ρ_0, *dependent on the physical parameters of equation (5), such that* $\limsup_{t\to\infty} |\boldsymbol{u}(t)| \leq \rho_0$.

Proof. We denote by $r_1 = \frac{1}{a}$, $r_2 = l_c$, and $r_3 = 4l_cB^2$. From (2), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|(K^{\frac{1}{2}}\varphi, \Phi, \Psi)|^2 &= -r_1\frac{\nu}{2}\|\varphi\|^2 + r_1 a(\psi_c(\Phi + \varphi), \varphi) + r_2\frac{1}{l_c}((\overline{\psi_c} - \Psi), \Phi)\\ &\quad + r_3\frac{1}{4l_cB^2}(\Phi, \Psi) - r_3\frac{1}{4l_cB^2}(F_T^{-1}\Psi, \Psi)\\ &= -\frac{\nu}{2a}\|\varphi\|^2 + (\psi_c(\Phi + \varphi), \Phi + \varphi) - (F_T^{-1}\Psi, \Psi).\end{aligned}$$

Here we used the fact that $\frac{\partial}{\partial\theta}$ is antisymmetric, and $\bar{\varphi} = 0$. Due to (3), there exist constants $C_1, C_2 > 0$ such that

$$\psi_c(\Phi(t) + \varphi(\theta, t))(\Phi(t) + \varphi(\theta, t)) \leq -C_1(\Phi(t) + \varphi(\theta, t))^2 + C_2.$$

Keeping in mind that K^{-1} is a bounded operator, there are some constants M_1, M_2, $M_3 > 0$ for which

$$\frac{1}{2}\frac{d}{dt}|K^{\frac{1}{2}}\boldsymbol{u}|^2 + \frac{\nu}{2a}\|\varphi\|^2 \leq -C_1|\Phi + \varphi|^2 - |\Psi|^{\frac{3}{2}} + C_2 \tag{9a}$$

$$\leq C_1(-|\Phi + \varphi|^{\frac{3}{2}} + 1) - |\Psi|^{\frac{3}{2}} + C_2 \tag{9b}$$

$$\le -(C_1^{\frac{4}{3}}|\Phi+\varphi|^2 + |\Psi|^2)^{\frac{3}{4}} + C_1 + C_2 \tag{9c}$$

$$\le -M_1(|\boldsymbol{u}|^2)^{\frac{3}{4}} + M_2 \le -M_3(|K^{\frac{1}{2}}\boldsymbol{u}|^2)^{\frac{3}{4}} + M_2. \tag{9d}$$

Now we let $y(t) = |K^{\frac{1}{2}}\boldsymbol{u}(t)|^2$ and from the above we have

$$\frac{dy}{dt} \le -2M_3\left(y^{\frac{3}{4}}(t) - \frac{M_2}{M_3}\right). \tag{10}$$

Since $y(t)$ is a continuous function (this is certainly true when we treat the Galerkin system), the above implies that $y \in L^\infty((0,T))$, for any $T > 0$, and thus,

$$\boldsymbol{u} \in L^\infty([0,T];\mathcal{H}). \tag{11}$$

Furthermore, one can easily show that

$$\limsup_{t\to\infty} |\boldsymbol{u}(t)|^2 \le \rho_0 := \|K^{-\frac{1}{2}}\|^2_{\mathcal{L}(\mathbb{H}\times\mathbb{R})}\left(\frac{M_2}{M_3}\right)^{\frac{4}{3}}. \tag{12}$$

That is, there is an absorbing ball in $\mathcal{H}$ of radius ρ_0.

Integrating (9d) over the interval $[0,T]$, we have

$$\frac{1}{T}\int_0^T \|\varphi(t)\|^2\,dt \le \frac{2a}{\nu}\left(M_2 + \frac{1}{2T}|\boldsymbol{u}_0|^2\right). \tag{13}$$

Since $\|\Phi\|_{H^1} = |\Phi|_{L^2} = |\Phi|$ and $\|\Psi\|_{H^1} = |\Psi|_{L^2} = |\Psi|$, then by (11), $\Phi, \Psi \in L^\infty([0,T];\mathbb{R})$ together with (13) implies $\boldsymbol{u} \in L^2([0,T];\mathcal{V})$.

Next we will prove that $\frac{d\boldsymbol{u}}{dt}$ belongs to $L^2([0,T];\mathcal{V}')$. Using the Sobolev imbedding theorem and interpolation inequalities in 1D, we have

$$\|\varphi\|_{L^6} \le C\|\varphi\|_{H^{\frac{1}{3}}} \le C'|\varphi|_{L^2}^{\frac{2}{3}}\|\varphi\|_{H^1}^{\frac{1}{3}}. \tag{14}$$

By Cauchy-Schwarz and by using (3), we find

$$\begin{aligned}
|\psi_c(\Phi(t)+\varphi(t)) - \overline{\psi_c}| &\le (1+2\pi)\|\psi_c(\Phi(t)+\varphi(t))\|_{L^2} && \text{(15a)}\\
&\le C_3||\varphi(t)+\Phi(t)||^3_{L^6} + C_4 && \text{(15b)}\\
&\le C_3(||\varphi(t)||_{L^6} + \sup_{0\le t\le T}||\Phi(t)||_{L^6})^3 + C_4, && \text{(15c)}\\
&\quad \text{using (11) and (14) we obtain} && \text{(15d)}\\
&\le C_5(|\varphi(t)|^2||\varphi(t)|| + 1)(1+T^2). && \text{(15e)}
\end{aligned}$$

Let $v \in \mathbb{V}$, then by taking the action of (2a) on v and using (11) and (15e), we have

$$\begin{aligned}
\left\langle K\frac{\partial\varphi}{\partial t}, v\right\rangle &= -\frac{\nu}{2}\langle A\varphi, v\rangle - \frac{1}{2}\left\langle\frac{\partial\varphi}{\partial\theta}, v\right\rangle + a\langle\psi_c(\Phi+\varphi) - \overline{\psi_c}, v\rangle\\
&= -\frac{\nu}{2}((\varphi, v)) + \frac{1}{2}\left(\varphi, \frac{\partial v}{\partial\theta}\right) + a(\psi_c(\Phi+\varphi) - \overline{\psi_c}, v)\\
&\le \frac{\nu}{2}\|\varphi\|\|v\| + \frac{1}{2}|\varphi|\|v\| + 2aC_5(|\varphi|^2\|\varphi\| + 1)(1+T^2)|v|\\
&\le C_6(\|\varphi\| + 1)(1+T^2)\|v\|,
\end{aligned}$$

which implies that $\|\frac{d\varphi}{dt}(t)\|_{\mathbb{V}'} \in L^2([0,T])$ since $\varphi \in L^2([0,T];\mathbb{V}) \cap L^\infty([0,T];\mathbb{H})$ and K^{-1} is bounded on $\mathbb{V}' \times \mathbb{R}$.

Φ and Ψ are continuous functions in t; hence we only need to prove weak continuity of $\varphi(t)$. Letting $v \in \mathbb{V}$ be arbitrary, we take the dual action of (2a) on v, and integrate over the interval $[t_0, t]$ to obtain

$$\begin{aligned}\langle\varphi(t), v\rangle - \langle\varphi(t_0), v\rangle + \frac{\nu}{2}\int_{t_0}^{t} ((K^{-1}\varphi(s), v))\,ds \\ = \frac{1}{2}\int_{t_0}^{t}\left(K^{-1}\varphi(s), \frac{dv}{d\theta}\right) ds + \int_{t_0}^{t} (aK^{-1}(\psi_c - \overline{\psi_c})(s), v)\,ds.\end{aligned}$$

Since $\varphi(t), \varphi(t_0) \in \mathbb{H}$ for almost every t and t_0, we substitute $\langle\varphi(t), v\rangle = (\varphi(t), v)$ and $\langle\varphi(t_0), v\rangle = (\varphi(t_0), v)$. Therefore, from the above and (15e), the weak continuity of $\varphi(t)$ in $\mathbb{H}$ follows because $\mathbb{V}$ is dense in $\mathbb{H}$ and $\varphi \in L^\infty([0,T];\mathbb{H}) \cap L^2([0,T];\mathbb{V})$. Next we will show the continuity of $|\boldsymbol{u}(t)|^2$, which together with the above implies the strong continuity of $\boldsymbol{u}(t)$ in $\mathcal{H}$. Indeed, let us recall that $\boldsymbol{u} \in L^2([0,T];\mathcal{V})$ and $\frac{d\boldsymbol{u}}{dt} \in L^2([0,T], \mathcal{V}')$. Then by Lemma 1.2 of Chapter 3 in [40] we have

$$2\left\langle\frac{d\boldsymbol{u}}{dt}(t), \boldsymbol{u}(t)\right\rangle = \frac{d}{dt}|\boldsymbol{u}(t)|^2, \quad \text{as distributions on } (0,T), \tag{16}$$

and the left-hand side of (16) is an $L^1([0,T])$ function. Therefore $\frac{d}{dt}|\boldsymbol{u}(t)|^2, |\boldsymbol{u}(t)|^2 \in L^1([0,T])$. As a result we have that $|\boldsymbol{u}(t)|^2$ is absolutely continuous.

To show the uniqueness of the weak solution, let $\boldsymbol{u}$ and $\boldsymbol{v}$ be any two weak solutions with the initial values $\boldsymbol{u}_0$ and $\boldsymbol{v}_0$, respectively. Since $\frac{d(\boldsymbol{u}-\boldsymbol{v})}{dt} \in L^2([0,T];\mathcal{V}')$ and $(\boldsymbol{u}-\boldsymbol{v}) \in L^2([0,T];\mathcal{V})$, for a.e. $t \in [0,T]$ we have

$$\left\langle\frac{d(\boldsymbol{u}-\boldsymbol{v})}{dt}, \boldsymbol{u}-\boldsymbol{v}\right\rangle = \langle \boldsymbol{L}(\boldsymbol{u}-\boldsymbol{v}), \boldsymbol{u}-\boldsymbol{v}\rangle + \langle \mathbf{f}(\boldsymbol{u}) - \mathbf{f}(\boldsymbol{v}), \boldsymbol{u}-\boldsymbol{v}\rangle.$$

Again by Lemma 1.2 of Chapter 3 in [40] and (6) it follows that

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}-\boldsymbol{v}|^2 + \frac{\nu}{2}|K^{-\frac{1}{2}}A^{\frac{1}{2}}(\varphi_{\boldsymbol{u}} - \varphi_{\boldsymbol{v}})|^2 \le |\mathbf{f}(\boldsymbol{u}) - \mathbf{f}(\boldsymbol{v})||\boldsymbol{u}-\boldsymbol{v}|, \tag{17}$$

and since $\mathbf{f}$ is differentiable, $\mathbf{f}(\boldsymbol{u}) - \mathbf{f}(\boldsymbol{v}) = \int_0^1 \mathbf{f}'(\eta\boldsymbol{u} + (1-\eta)\boldsymbol{v})(\boldsymbol{u}-\boldsymbol{v})\,d\eta$. Hence,

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}-\boldsymbol{v}|^2 + \frac{\nu}{2}|K^{-\frac{1}{2}}A^{\frac{1}{2}}(\varphi_{\boldsymbol{u}} - \varphi_{\boldsymbol{v}})|^2 \le \left(\int_0^1 \|\mathbf{f}'(\eta\boldsymbol{u} + (1-\eta)\boldsymbol{v})\|_{L^\infty(\Omega)}\,d\eta\right)|\boldsymbol{u}-\boldsymbol{v}|^2.$$

By virtue of (5) and (3) there exist some positive constants C_7 and C_8 such that

$$\|\mathbf{f}'(\boldsymbol{w})\|_{L^\infty(\Omega)} \le C_7\|\boldsymbol{w}\|^2_{L^\infty(\Omega)} + C_8.$$

Applying Agmon's inequality, $\|\boldsymbol{w}\|^2_{L^\infty(\Omega)} \le |\boldsymbol{w}|\|\boldsymbol{w}\|$, we reach

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}-\boldsymbol{v}|^2 + \frac{\nu}{2}|K^{-\frac{1}{2}}A^{\frac{1}{2}}(\varphi_{\boldsymbol{u}} - \varphi_{\boldsymbol{v}})|^2 \le (2C_7(|\boldsymbol{u}|\|\boldsymbol{u}\| + |\boldsymbol{v}|\|\boldsymbol{v}\|) + C_8)|\boldsymbol{u}-\boldsymbol{v}|^2. \tag{18}$$

Thanks to (11) and (13), $\boldsymbol{u}, \boldsymbol{v} \in L^\infty([0,T];\mathcal{H}) \cap L^2([0,T];\mathcal{V})$, and therefore

$$\mu(t) := \int_0^t (2C_7(|\boldsymbol{u}|\|\boldsymbol{u}\| + |\boldsymbol{v}|\|\boldsymbol{v}\|) + C_8)\, dt \le \mu(T) < \infty, \quad \text{for } t \in [0,T]. \tag{19}$$

Applying Gronwall's and Cauchy-Schwarz inequalities to (18) gives

$$|\boldsymbol{u}(t) - \boldsymbol{v}(t)|^2 \le |\boldsymbol{u}_0 - \boldsymbol{v}_0|^2 e^{2\mu(t)}. \tag{20}$$

This implies that the solution is unique when $\boldsymbol{u}_0 = \boldsymbol{v}_0$ and continuously depends on the initial data, i.e., if $\boldsymbol{u}_0 \to \boldsymbol{v}_0$ in $\mathcal{H}$, then $\boldsymbol{u}(t) \to \boldsymbol{v}(t)$ in $L^\infty([0,T];\mathcal{H})$. □

Corollary 2. *Let $T > 0$ and $\boldsymbol{u}_0 \in \mathcal{H}$ be given, and let $\boldsymbol{u}(\theta,t)$ be the corresponding unique weak solution of the system (5). Then $\boldsymbol{u} \in L^\infty_{loc}((0,T],\mathcal{V})$. Furthermore, there exists a constant ρ_1, dependent on the physical parameters of equation (5), but independent of the initial data, such that $\limsup_{t\to\infty} \|\boldsymbol{u}(t)\| \le \rho_1$. That is, $S(t)\boldsymbol{u}_0$ has an absorbing ball in $\mathcal{V}$.*

Proof. Here again we present a formal proof. To provide a more rigorous proof, one should establish the following estimates first for the Galerkin approximating system and then pass to the limit using the appropriate compactness theorems.

Due to the antisymmetricity of $\frac{\partial}{\partial\theta}$ and the periodicity of the solutions, (2) implies

$$\frac{1}{2}\frac{d}{dt}\|K^{\frac{1}{2}}\varphi\|^2 + \frac{\nu}{2}|A\varphi|^2 = a\left(\frac{\partial}{\partial\theta}\psi_c(\Phi+\varphi), \frac{\partial}{\partial\theta}(\Phi+\varphi)\right).$$

Here we note that $\frac{\partial}{\partial\theta}\psi_c(\Phi+\varphi) = g(\Phi+\varphi)\frac{\partial\varphi}{\partial\theta}$ where $g(x) = \psi_c'(x)$. Due to (3), $g(x)$ is bounded from above, and we have

$$\frac{1}{2}\frac{d}{dt}\|K^{\frac{1}{2}}\varphi\|^2 + \frac{\nu}{2}|A\varphi|^2 \le C\|\Phi+\varphi\|^2. \tag{21}$$

By using the fact that Φ is independent of θ and K^{-1} is bounded, we have

$$\frac{1}{2}\frac{d}{dt}\|K^{\frac{1}{2}}\varphi\|^2 + \frac{\nu}{2}|A\varphi|^2 \le C_9\|K^{\frac{1}{2}}\varphi\|^2. \tag{22}$$

Let $y(t) = \|K^{\frac{1}{2}}\varphi(t)\|^2$, then it follows that

$$y(t) \le e^{2C_9(t-s)}y(s) \le e^{2C_9T}y(s), \quad \text{for } 0 \le s \le t \le T.$$

Let $t > 0$ be fixed. We integrate the above inequality with respect to s over the interval $(0,t)$ and use (13) to obtain the following inequality:

$$\begin{aligned}\int_0^t y(t)\,ds &\le e^{2C_9T}\int_0^t y(s)\,ds \le e^{2C_9T}\int_0^T y(s)\,ds \\ &\le e^{2C_9T}\frac{2a}{\nu}\|K^{\frac{1}{2}}\|^2_{\mathcal{L}(\mathbb{V})}\left(M_2T + \frac{1}{2}|\boldsymbol{u}_0|^2\right).\end{aligned}$$

This implies

$$y(t) \leq \frac{e^{2C_9 T} a(2M_2 T + |\boldsymbol{u}_0|^2)\|K^{\frac{1}{2}}\|^2_{\mathcal{L}(\mathbb{V})}}{\nu t}, \quad \text{for all } \ t \in (0, T]. \tag{23}$$

Since $K^{-\frac{1}{2}}$ is bounded, we conclude that $\boldsymbol{u} \in L^\infty_{\text{loc}}((0, T]; \mathcal{V})$ whenever $\boldsymbol{u}_0 \in \mathcal{H}$.

In order to prove the rest of the Corollary, we let $\boldsymbol{u}_0 \in \mathcal{H}$ and take $T = 2$ in (23). Hence from (23), we have

$$y(t) \leq \frac{e^{4C_9} a(4M_2 + |\boldsymbol{u}_0|^2)\|K^{\frac{1}{2}}\|^2_{\mathcal{L}(\mathbb{V})}}{\nu t}, \quad \text{for } 0 < t \leq 2. \tag{24}$$

However, for $t > 1$, following inequality (22), we have

$$\frac{d}{dt}\|K^{\frac{1}{2}}\varphi\|^2 \leq 2C_9\|K^{\frac{1}{2}}\varphi\|^2.$$

Hence we obtain

$$\begin{aligned} \|K^{\frac{1}{2}}\varphi(t)\|^2 &\leq e^{2C_9(t-s)}\|K^{\frac{1}{2}}\varphi(s)\|^2 \\ &\leq e^{2C_9}\|K^{\frac{1}{2}}\varphi(s)\|^2, \quad \forall s, t \ \text{ such that } \ t-1 \leq s \leq t, \end{aligned}$$

provided $t \geq 1$. In particular, when we integrate the above with respect to s over the interval $[t-1, t]$, we have

$$\|K^{\frac{1}{2}}\varphi(t)\|^2 \leq e^{2C_9}\int_{t-1}^{t}\|K^{\frac{1}{2}}\varphi(s)\|^2\, ds, \quad \text{for all } t \geq 1. \tag{25}$$

By using (13), it follows that

$$\int_{t-1}^{t}\|K^{\frac{1}{2}}\varphi(s)\|^2\, ds \leq \frac{2a}{\nu}\left(M_2 + \frac{1}{2}|\boldsymbol{u}(t-1)|^2\right)\|K^{\frac{1}{2}}\|^2_{\mathcal{L}(\mathbb{V})}. \tag{26}$$

By (25), (26), and Theorem 1 we have

$$\limsup_{t\to\infty}\|K^{\frac{1}{2}}\varphi(t)\|^2 \leq e^{2C_9}\left(\frac{2a}{\nu}M_2 + \frac{a}{\nu}\rho_0^2\right)\|K^{\frac{1}{2}}\|^2_{\mathcal{L}(\mathbb{V})}. \tag{27}$$

From (12) and the above,

$$\begin{aligned} \limsup_{t\to\infty}\|\boldsymbol{u}(t)\|^2_{\mathcal{V}} &\leq 2(\limsup_{t\to\infty}\|\varphi(t)\|^2 + \limsup_{t\to\infty}(|\Phi(t)| + |\Psi(t)|)^2) \\ &\leq 2\|K^{-\frac{1}{2}}\|^2_{\mathcal{L}(\mathbb{V})}\|K^{\frac{1}{2}}\|^2_{\mathcal{L}(\mathbb{V})}e^{2C_9}\left(\frac{2a}{\nu}M_2 + \frac{a}{\nu}\rho_0^2\right) + 2M_4\rho_0^2 =: \rho_1. \end{aligned}$$

Here, the positive constant M_4 depends only on the given constants r_2 and r_3. □

Theorem 3. *Let* $T > 0$ *and* $\boldsymbol{u}_0 \in \mathcal{V}$ *be given. Then there exists a unique strong solution of system (5) on the interval* $[0, T]$.

Proof. Since $\boldsymbol{u}_0 \in \mathcal{V} \subset \mathcal{H}$, due to Theorem 1, system (5) has a unique weak solution $\boldsymbol{u}$ which, thanks to (11) and (21), satisfies $\boldsymbol{u} \in L^\infty([0,T];\mathcal{H})$. The rest of the proof is similar to the proof of Theorem 1, where we replace the $\mathcal{V}'$ action by $\mathcal{H}$ inner product, which is easy to justify in this case. Furthermore, to obtain higher regularity, we use (22) and the fact that $\boldsymbol{u}_0 \in \mathcal{V}$. For more details see [5].

Clearly, each strong solution is the weak solution, and hence the uniqueness of the strong solution follows by Theorem 1 (from the uniqueness of weak solutions). $\square$

Remark 3. From the uniqueness of weak solutions in Theorem 1 and Corollary 2, we have $\boldsymbol{u}(t) \in \mathcal{V}$ for almost every $t > 0$, even if $\boldsymbol{u}_0 = \boldsymbol{u}(t=0) \in \mathcal{H}$. Therefore, by Theorem 3, the weak solution becomes instantaneously a strong solution.

The following theorem contains essential properties of the semigroup solution operator to system (5). For instance, the differentiability of the operator $S(t)$ with respect to the initial data is necessary to estimate the dimension of a global attractor [41], a property which is not often emphasized in literature. Also, the existence of an absorbing ball in $D(A) \times \mathbb{R} \times \mathbb{R}$ is used in proving the existence of finite-dimensional IM.

Theorem 4. *Let $\boldsymbol{u}_0 \in \mathcal{H}$ and $T > 0$ be given. Then the solution of the system (5), $\boldsymbol{u} \in L^2_{loc}((0,T];D(A) \times \mathbb{R} \times \mathbb{R})$. Furthermore, for every $t > 0$ fixed, the semigroup solution operator $S(t)\colon \mathcal{H} \to D(A) \times \mathbb{R} \times \mathbb{R}$ is continuous and Fréchet differentiable, with respect to the initial data, in $\mathcal{H}$. Moreover, there exists a positive constant ρ_2 such that (28) holds.*

Proof. Following steps similar to those used in Theorem 1 and Corollary 2 and taking advantage of the nice structure of the nonlinearity, one can easily show that $\boldsymbol{u} \in L^2_{\mathrm{loc}}((0,T];D(A) \times \mathbb{R} \times \mathbb{R})$, and by analogy to (25) and (26), one can show that

$$|A\varphi(t)|^2 \le M_5 \int_{t-1}^{t} |A\varphi(s)|^2\,ds \le M_5(M_6 + M_7\|\boldsymbol{u}(t-1)\|^2).$$

Hence, by the above and Corollary 2, we have

$$\limsup_{t\to\infty} |A\varphi(t)|^2 \le M_5(M_6 + M_7\rho_1) =: \rho_2. \tag{28}$$

Following the same steps used in Corollary 2, we can also show that the solution $S(t)\colon \mathcal{H} \to D(A) \times \mathbb{R} \times \mathbb{R}$ is continuous for fixed time $t > 0$.

For the sake of completeness we present here the differentiability of the semigroup with respect to the initial data. For given weak solutions $\boldsymbol{u}, \boldsymbol{v}$, the difference $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$ satisfies

$$\frac{d\boldsymbol{w}}{dt} = \boldsymbol{L}\boldsymbol{w} + \mathbf{f}(\boldsymbol{u}) - \mathbf{f}(\boldsymbol{v}), \quad \text{in } \ L^2([0,T];\mathcal{V}'). \tag{29}$$

Using (18), (19), and (20) it also follows that

$$\int_0^t \|\boldsymbol{u}(t) - \boldsymbol{v}(t)\|^2 \le M_8|\boldsymbol{u}_0 - \boldsymbol{v}_0|^2(1 + e^{\mu(T)}\mu(T)T), \quad \text{for } t \in [0,T]. \tag{30}$$

By using (3) we rewrite $\mathbf{f}(\boldsymbol{u}) - \mathbf{f}(\boldsymbol{v}) = l_0(t)\boldsymbol{w} + l_1(t, \boldsymbol{w})$, where

$$\begin{aligned} l_0(t)\boldsymbol{w} \;=\; & (aK^{-1}(\psi_c'(\Phi_{\boldsymbol{u}} + \varphi_{\boldsymbol{u}})(\Phi_{\boldsymbol{w}} + \varphi_{\boldsymbol{w}}) - \overline{\psi_c'(\Phi_{\boldsymbol{u}} + \varphi_{\boldsymbol{u}})(\Phi_{\boldsymbol{w}} + \varphi_{\boldsymbol{w}})}), \\ & {l_c}^{-1}(\overline{\psi_c'(\Phi_{\boldsymbol{u}} + \varphi_{\boldsymbol{u}})(\Phi_{\boldsymbol{w}} + \varphi_{\boldsymbol{w}})} - \Psi_{\boldsymbol{w}}), (4l_cB^2)^{-1}(\Phi_{\boldsymbol{w}} - (F_T^{-1})'(\Psi_{\boldsymbol{u}})\Psi_{\boldsymbol{w}})), \end{aligned}$$

and

$$\begin{aligned} l_1(t, \boldsymbol{w}) \;=\; & (aK^{-1}((-k_0(\Phi_{\boldsymbol{w}} + \varphi_{\boldsymbol{w}})^2((\Phi_{\boldsymbol{w}} + \varphi_{\boldsymbol{w}}) - 3(\Phi_{\boldsymbol{u}} + \varphi_{\boldsymbol{u}})) - k_1(\Phi_{\boldsymbol{w}} + \varphi_{\boldsymbol{w}})^2) \\ & - (-k_0\overline{(\Phi_{\boldsymbol{w}} + \varphi_{\boldsymbol{w}})^2}(\overline{(\Phi_{\boldsymbol{w}} + \varphi_{\boldsymbol{w}})} - 3\overline{(\Phi_{\boldsymbol{u}} + \varphi_{\boldsymbol{u}})}) - k_1\overline{(\Phi_{\boldsymbol{w}} + \varphi_{\boldsymbol{w}})^2})), \\ & {l_c}^{-1}(-k_0\overline{(\Phi_{\boldsymbol{w}} + \varphi_{\boldsymbol{w}})^2}(\overline{(\Phi_{\boldsymbol{w}} + \varphi_{\boldsymbol{w}})} - 3\overline{(\Phi_{\boldsymbol{u}} + \varphi_{\boldsymbol{u}})}) - k_1\overline{(\Phi_{\boldsymbol{w}} + \varphi_{\boldsymbol{w}})^2}), \\ & (4l_cB^2)^{-1}(-(F_T^{-1}(\Psi_{\boldsymbol{u}}) - F_T^{-1}(\Psi_{\boldsymbol{v}})) + (F_T^{-1})'(\Psi_{\boldsymbol{u}})\Psi_{\boldsymbol{w}})). \end{aligned}$$

We recall (23) and Remark 3. Thus, in particular, we have $\boldsymbol{u} \in L^\infty_{\mathrm{loc}}((0, T]; \mathcal{V})$. Here we find, by using Agmon's inequality and the structure of $l_0(t)$,

$$\begin{aligned} |l_0(t)\boldsymbol{w}|_{\mathcal{H}} \;\le\; & \left(\int_\Omega c_0(|\boldsymbol{u}(t, \theta)|^4 + 1)|\boldsymbol{w}(t, \theta)|^2 \, d\theta\right)^{\frac{1}{2}} && \text{(31a)} \\ \le\; & c_1(\|\boldsymbol{u}\|^2_{L^\infty(\Omega)} + 1)|\boldsymbol{w}| && \text{(31b)} \\ \le\; & c_2(|\boldsymbol{u}|\|\boldsymbol{u}\| + 1)|\boldsymbol{w}|. && \text{(31c)} \end{aligned}$$

Moreover, thanks to the smoothing effect, for any given weak solutions $\boldsymbol{u}, \boldsymbol{v}$ and for every $t > 0$, the linear operator $l_0(t) \in \mathcal{L}(\mathcal{H})$.

Using the Sobolev imbedding theorem and interpolation inequalities in 1D, we also find

$$\begin{aligned} |\mathbf{f}(\boldsymbol{u}) - \mathbf{f}(\boldsymbol{v}) - l_0(t)\boldsymbol{w}|_{\mathcal{H}} \;\le\; & \left(\int_\Omega c_3|\boldsymbol{w}(t, \theta)|^4(|\boldsymbol{w}(t, \theta)|^2 + |\boldsymbol{u}(t, \theta)|^2 + 1) \, d\theta\right)^{\frac{1}{2}} && \text{(32a)} \\ \le\; & c_3(\|\boldsymbol{w}\|^3_{L^6} + (\|\boldsymbol{u}\|_{L^\infty(\Omega)} + 1)\|\boldsymbol{w}\|^2_{L^4}) && \text{(32b)} \\ \le\; & c_4(\|\boldsymbol{w}\|^3_{H^{\frac{1}{3}}} + (|\boldsymbol{u}|^{\frac{1}{2}}\|\boldsymbol{u}\|^{\frac{1}{2}} + 1)\|\boldsymbol{w}\|^2_{H^{\frac{1}{4}}}) && \text{(32c)} \\ \le\; & c_5(|\boldsymbol{w}|^2\|\boldsymbol{w}\| + (|\boldsymbol{u}|^{\frac{1}{2}}\|\boldsymbol{u}\|^{\frac{1}{2}} + 1)|\boldsymbol{w}|^{\frac{3}{2}}\|\boldsymbol{w}\|^{\frac{1}{2}}). && \text{(32d)} \end{aligned}$$

Now we consider the formal "linearized equation" around the solution $\boldsymbol{u}(t)$,

$$\frac{dU}{dt} = \boldsymbol{L}U + l_0(t)U, \qquad U(0) = U_0. \tag{33}$$

By following the proofs of Theorem 1 and Theorem 3, we can easily prove that there exist unique weak and strong solutions U for the differential equation (33) provided $U_0 \in \mathcal{H}$ or $\mathcal{V}$, respectively.

Let $U_0 = \boldsymbol{w}_0 = \boldsymbol{u}_0 - \boldsymbol{v}_0$ and subtract equation (33) from (29). We denote $\xi = \boldsymbol{u} - \boldsymbol{v} - U = \boldsymbol{w} - U$; then ξ satisfies

$$\frac{d\xi}{dt} = \boldsymbol{L}\xi + l_0(t)\xi + l_1(t, \boldsymbol{w}), \qquad \xi(0) = 0. \tag{34}$$

For weak solutions $\boldsymbol{u}$, $\boldsymbol{v}$, and U, the above equation holds in $L^2([0,T];\mathcal{V}')$. Thus, since $\boldsymbol{u},\boldsymbol{v}\in L^2([0,T];\mathcal{V})$, taking the dual action of the above equation on ξ, and using Lemma 1.2 of Chapter 3 in [40] with (20), (31c), (32d), and Young's inequality, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|\xi|^2+\frac{\nu}{2}\|K^{-\frac{1}{2}}\xi\|^2 &\le |l_0(t)\xi|\,|\xi|+|l_1(t,\boldsymbol{w})|\,|\xi| &&(35\text{a})\\
&\le \left(c_2|\boldsymbol{u}|\|\boldsymbol{u}\|+c_2+\frac{c_5^2}{2}\right)|\xi|^2\\
&\quad+\frac{1}{2}(|\boldsymbol{w}|^4\|\boldsymbol{w}\|^2+(|\boldsymbol{u}|\|\boldsymbol{u}\|+1)|\boldsymbol{w}|^3\|\boldsymbol{w}\|) &&(35\text{b})\\
&\le \left(c_6(1+T)\|\boldsymbol{u}\|+c_2+\frac{c_5^2}{2}\right)|\xi|^2+\frac{1}{2}e^{2\mu(T)}|\boldsymbol{w}_0|^4\|\boldsymbol{w}\|^2 &&(35\text{c})\\
&\quad+(c_7(1+T)\|\boldsymbol{u}\|+1)\frac{1}{2}e^{\frac{3}{2}\mu(T)}|\boldsymbol{w}_0|^3\|\boldsymbol{w}\|.
\end{aligned}$$

From the fact that the weak solutions $\boldsymbol{u},\boldsymbol{v}\in L^2([0,T];\mathcal{V})$ we also find

$$C(t):=\int_0^t(2c_6(1+T)\|\boldsymbol{u}(s)\|+2c_2+c_5^2)\,ds\le C(T)<\infty,\quad \text{for } t\in(0,T].$$

Therefore, applying Gronwall's inequality to (35) and using (30) gives

$$\begin{aligned}
|\xi|^2 &\le c_8e^{C(T)}\left(e^{2\mu(T)}|\boldsymbol{w}_0|^4\int_0^t\|\boldsymbol{w}\|^2\,dt+e^{\frac{3}{2}\mu(T)}|\boldsymbol{w}_0|^3\int_0^t(\|\boldsymbol{u}\|+1)\|\boldsymbol{w}\|\,dt\right)\\
&\le c_8e^{C(T)}\Bigg(e^{2\mu(T)}M_8(1+e^{\mu(T)}\mu(T)T)|\boldsymbol{w}_0|^6\\
&\qquad+e^{\frac{3}{2}\mu(T)}|\boldsymbol{w}_0|^3\int_0^t(\|\boldsymbol{u}\|+1)\|\boldsymbol{w}\|\,dt\Bigg),\quad \text{for } t\in(0,T].
\end{aligned}$$

Again by using $\boldsymbol{u}\in L^2([0,T];\mathcal{V})$, (30), and the Cauchy-Schwarz inequality, we find

$$\int_0^t(\|\boldsymbol{u}(s)\|+1)\|\boldsymbol{w}(s)\|\,ds\le\tilde{C}(T)(M_8(1+e^{\mu(T)}\mu(T)T))^{\frac{1}{2}}|\boldsymbol{w}_0|,$$

for $t\in(0,T]$, where $\tilde{C}(T)$ is a positive constant depending only on T.

Finally we conclude that

$$\begin{aligned}
\frac{|\boldsymbol{u}(t)-\boldsymbol{v}(t)-U(t)|^2}{|\boldsymbol{u}_0-\boldsymbol{v}_0|^2} &\le c_8e^{C(T)}e^{2\mu(T)}M_8(1+e^{\mu(T)}\mu(T)T)|\boldsymbol{u}_0-\boldsymbol{v}_0|^4\\
&\quad+c_8e^{C(T)}e^{\frac{3}{2}\mu(T)}\tilde{C}(T)(M_8(1+e^{\mu(T)}\mu(T)T))^{\frac{1}{2}}|\boldsymbol{u}_0-\boldsymbol{v}_0|^2\to0,
\end{aligned}$$

as $\boldsymbol{v}_0\to\boldsymbol{u}_0$ in $\mathcal{H}$, for $t\in(0,T]$. This proves that the mapping $S(t)\colon\boldsymbol{u}_0\to\boldsymbol{u}(t)$ is differentiable from $\mathcal{H}$ into $\mathcal{H}$, for $t\in(0,T]$. □

So far we have shown the existence of absorbing balls in $\mathcal{H}$, $\mathcal{V}$, $D(A)$, and the differentiability of semigroup $S(t)$ with respect to the initial data. Since there is an absorbing set $\mathcal{B}_{\mathcal{V}}$ in $\mathcal{V}$ for the semigroup $S(t)$, it follows that $S(t)\mathcal{B} \subset \mathcal{B}_{\mathcal{V}}$ for t large, where $\mathcal{B}$ is any bounded set of $\mathcal{H}$. Notice that $\mathcal{V} \subset \mathcal{H}$ and that the inclusion map is compact, i.e., $\mathcal{V}$ is compactly embedded in $\mathcal{H}$. Thus $S(t)\mathcal{B}$ is precompact in $\mathcal{H}$, i.e., $\overline{S(t)\mathcal{B}}$ is compact in $\mathcal{H}$ where the closure is taken in the $\mathcal{H}$ topology. Finally by Theorem 1.1 of Chapter 1 in [41], we conclude that there exists a global attractor

$$\mathcal{A} = \bigcap_{\tau>0} \overline{\bigcup_{t\geq\tau} S(t)\mathcal{B}_{\mathcal{H}}},$$

which is compact and connected and attracts the bounded sets of $\mathcal{H}$. Here we notice that $\mathcal{A}$ is nonempty since the sets $\overline{\bigcup_{t\geq\tau} S(t)\mathcal{B}_{\mathcal{H}}}$ are nonempty, decreasing as τ increases, and compact in $\mathcal{H}$.

These are preliminary properties which are needed for estimating the dimension of the global attractor. We refer the reader to [41] for more details about estimating the dimension of an attractor and to [3] where the authors presented upper bounds of the fractal and Hausdorff dimensions of the global attractor for the underlying equation.

5. Gevrey Regularity

In this section we enhance the regularity results in Section 4. We prove that the solution of the underlying PDE becomes instantaneously spatially analytic. Moreover, we show the fast (exponential) decay of Fourier coefficients, which is a consequence of a special Gevrey class of regularity (for similar results, see [10] and [11]). As a result this justifies some of the numerical results reported in [31], [32], and [34].

Here we choose to utilize the techniques from Foias and Temam (see [12] and [15]) to establish the Gevrey regularity of our system of equation (5). As it has been observed in [12], to establish such a regularity result, it is important that the given PDE has analytic nonlinear terms. In our equation, the nonlinear terms are analytic in the spatial variable θ because $\Phi(t)$ and $\Psi(t)$ only depend on the time variable t. However, unlike spatial analyticity, we are unable to prove the time analyticity of the solution with the usual tools. This is because the nonlinear term $F_T^{-1}(\Psi)$ is not an analytic function near $\Psi = 0$, and thus, $\Psi(t)$ may not be analytic in t.

Before we show the regularity, we remark again that we can justify this rigorously by considering the Galerkin approximation first and then passing to the limit.

For $\sigma > 0$ given, we consider the following Gevrey class:

$$G_\sigma = \left\{ u = \sum_{j\in\mathbb{Z}\setminus\{0\}} \hat{u}_j e^{ij\theta} : \sum_{j\in\mathbb{Z}\setminus\{0\}} |\hat{u}_j|^2 e^{2\sigma|j|} < \infty \right\}.$$

We note that $G_\sigma = D(e^{\sigma A^{\frac{1}{2}}})$, a Hilbert space with the inner product

$$(u, v)_{G_\sigma} = \sum_{j\in\mathbb{Z}\setminus\{0\}} \hat{u}_j \hat{v}_j^* \, e^{2\sigma|j|}, \quad \text{for every } u, v \in D(e^{\sigma A^{\frac{1}{2}}}),$$

with the corresponding norm

$$\|u\|_{G_\sigma} = \left(\sum_{j\in\mathbb{Z}\setminus\{0\}} |\hat{u}_j|^2 e^{2\sigma|j|} \right)^{1/2} .$$

We remark that this definition for the Gevrey class is slightly different from that introduced in [12] and [15] (see, however, [10], [24], [25], [39]). The proof of the following Lemma uses ideas from [10], [12], and [15] (see also [39]).

Lemma 5. *Let $t \geq 0$, and $u \in D(Ae^{tA^{\frac{1}{2}}})$, then*

$$\begin{aligned}
\left| \left(e^{tA^{\frac{1}{2}}} \left(u^3 - \frac{1}{2\pi}\int_0^{2\pi} u^3(\theta)\, d\theta \right), e^{tA^{\frac{1}{2}}} u \right) \right| &\leq C_1 |e^{tA^{\frac{1}{2}}} u|^3 \, \|e^{tA^{\frac{1}{2}}} u\| \\
&\leq c_1 |e^{tA^{\frac{1}{2}}} u|^{\frac{7}{2}} \, |Ae^{tA^{\frac{1}{2}}} u|^{\frac{1}{2}}, \\
\left| \left(e^{tA^{\frac{1}{2}}} \left(u^2 - \frac{1}{2\pi}\int_0^{2\pi} u^2(\theta)\, d\theta \right), e^{tA^{\frac{1}{2}}} u \right) \right| &\leq C_2 |e^{tA^{\frac{1}{2}}} u|^2 \, \|e^{tA^{\frac{1}{2}}} u\| \\
&\leq c_2 |e^{tA^{\frac{1}{2}}} u|^{\frac{5}{2}} \, |Ae^{tA^{\frac{1}{2}}} u|^{\frac{1}{2}},
\end{aligned}$$

for some positive constants C_1, c_1, C_2, and c_2, which are independent of u.

Proof. Let $u = \sum_{j\in\mathbb{Z}\setminus\{0\}} \hat{u}_j e^{ij\theta}$ and $\tilde{u} = \sum_{j\in\mathbb{Z}\setminus\{0\}} |\hat{u}_j| e^{ij\theta}$. Note that

$$\widehat{(u^3)}_k = \sum_{l+j=k} \left(\sum_{m+n=l} \hat{u}_m \hat{u}_n \right) \hat{u}_j ,$$

where $\widehat{(u^3)}_k$ is the k-th Fourier coefficient of u^3. Hence, we have

$$\begin{aligned}
&\left| \left(e^{tA^{\frac{1}{2}}} \left(u^3 - \frac{1}{2\pi}\int_0^{2\pi} u^3(\theta)\, d\theta \right), e^{tA^{\frac{1}{2}}} u \right) \right| \\
&\quad = \left| \sum_{k\in\mathbb{Z}\setminus\{0\}} e^{t|k|} \widehat{(u^3)}_k e^{t|k|} (\hat{u}_k)^* \right| \\
&\quad = \sum_{k\in\mathbb{Z}\setminus\{0\}} e^{t|k|} \left| \sum_{l+j=k} \left(\sum_{m+n=l} \hat{u}_m \hat{u}_n \right) \hat{u}_j \right| \; e^{t|k|} |\hat{u}_k| \\
&\quad \leq \sum_{k\in\mathbb{Z}\setminus\{0\}} \sum_{l+j=k} \sum_{m+n=l} e^{t|m|} |\hat{u}_m| e^{t|n|} |\hat{u}_n| e^{t|j|} |\hat{u}_j| e^{t|k|} |\hat{u}_k| \\
&\quad \leq \int |e^{tA^{\frac{1}{2}}} \tilde{u}(\theta, t)|^4 \, d\theta \leq \|e^{tA^{\frac{1}{2}}} \tilde{u}\|_{L^\infty}^2 |e^{tA^{\frac{1}{2}}} \tilde{u}|^2 .
\end{aligned}$$

Recall Agmon's inequality, $\|e^{tA^{\frac{1}{2}}}\tilde{u}\|_{L^\infty} \leq C_1|e^{tA^{\frac{1}{2}}}\tilde{u}|^{\frac{1}{2}}\|e^{tA^{\frac{1}{2}}}\tilde{u}\|^{\frac{1}{2}}$, and notice that $C_1|e^{tA^{\frac{1}{2}}}\tilde{u}| = c_1|e^{tA^{\frac{1}{2}}}u|$, then by interpolation inequality, we find

$$\begin{aligned}\left|\left(e^{tA^{\frac{1}{2}}}\left(u^3 - \frac{1}{2\pi}\int_0^{2\pi} u^3(\theta)\, d\theta\right), e^{tA^{\frac{1}{2}}}u\right)\right| &\leq c_1|e^{tA^{\frac{1}{2}}}u|^3\|e^{tA^{\frac{1}{2}}}u\| \\ &\leq c_1|e^{tA^{\frac{1}{2}}}u|^{\frac{7}{2}}|Ae^{tA^{\frac{1}{2}}}u|^{\frac{1}{2}}.\end{aligned}$$

The estimate for $|(e^{tA^{\frac{1}{2}}}(u^2 - \frac{1}{2\pi}\int_0^{2\pi} u^2(\theta)\, d\theta), e^{tA^{\frac{1}{2}}}u)|$ can be proved similarly. □

Theorem 6. *Let $R_0 > 0$, and $\boldsymbol{u}_0 \in \mathcal{H}$ such that $|\boldsymbol{u}_0|_{\mathcal{H}} \leq R_0$. Then there exists a $T^* > 0$, depending on R_0 and the other physical parameters of the system (2), in particular ν, such that $\boldsymbol{u}(\cdot, t) \in G_t$ for $t \in [0, T^*)$.*

Proof. Since the Gevrey regularity is a spatial regularity, we will need to consider only the unknown function $\varphi(\theta, t)$.

As we mentioned before, we present here a formal proof. A more rigorous proof can be established using the Galerkin procedure.

We first take the $\mathbb{H}$ inner product of the equation (2a) with $e^{2tA^{\frac{1}{2}}}\varphi$. Observe that $(\frac{\partial\varphi}{\partial\theta}, e^{2tA^{\frac{1}{2}}}\varphi) = (\frac{\partial}{\partial\theta}(e^{tA^{\frac{1}{2}}}\varphi), e^{tA^{\frac{1}{2}}}\varphi) = 0$. Hence by Cauchy-Schwarz and Young's inequalities, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|K^{\frac{1}{2}}e^{tA^{\frac{1}{2}}}\varphi|^2 &= (K^{\frac{1}{2}}A^{\frac{1}{2}}e^{tA^{\frac{1}{2}}}\varphi, K^{\frac{1}{2}}e^{tA^{\frac{1}{2}}}\varphi) \\ &\quad - \frac{\nu}{2}|A^{\frac{1}{2}}e^{tA^{\frac{1}{2}}}\varphi|^2 + (ae^{tA^{\frac{1}{2}}}(\psi_c(\Phi + \varphi) - \overline{\psi_c}), e^{tA^{\frac{1}{2}}}\varphi) \\ &\leq \left(\frac{\nu}{4} - \frac{\nu}{2}\right)|A^{\frac{1}{2}}e^{tA^{\frac{1}{2}}}\varphi|^2 + \frac{1}{\nu}|Ke^{tA^{\frac{1}{2}}}\varphi|^2 \\ &\quad + (ae^{tA^{\frac{1}{2}}}(\psi_c(\Phi + \varphi) - \overline{\psi_c}), e^{tA^{\frac{1}{2}}}\varphi).\end{aligned}$$

Notice that $\|\varphi(\cdot, t)\|_{G_t}$ is continuous in t; therefore there exists $\bar{T} > 0$ such that $\|\varphi\|_{G_t} + 1 \leq 2(\|\varphi_0\|_{G_0} + 1)$ for all t $\in [0, \bar{T})$. We also note that $\|\cdot\|_{G_0} = \|\cdot\|_{\mathbb{H}}$.

Imposing these inequalities, we will now find a lower estimate for $\bar{T}$. By Lemma 5 and using Young's inequality with (3), it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|K^{\frac{1}{2}}e^{tA^{\frac{1}{2}}}\varphi|^2 &\leq -\frac{\nu}{8}|A^{\frac{1}{2}}e^{tA^{\frac{1}{2}}}\varphi|^2 + \frac{1}{\nu}\|K^{\frac{1}{2}}\|^2_{\mathcal{L}(\mathbb{H})}|K^{\frac{1}{2}}e^{tA^{\frac{1}{2}}}\varphi|^2 \\ &\quad + (ak_0c_1)^2\frac{4}{\nu}|e^{tA^{\frac{1}{2}}}\varphi|^7 + (aC_3(\sup_{0\leq t<\bar{T}}|\Phi(t)| + 1))^2\frac{4}{\nu}|e^{tA^{\frac{1}{2}}}\varphi|^5 \\ &\quad + aC_4(\sup_{0\leq t<\bar{T}}|\Phi(t)|^2 + \sup_{0\leq t<\bar{T}}|\Phi(t)| + 1)|e^{tA^{\frac{1}{2}}}\varphi|^2.\end{aligned}$$

Let $y = |K^{\frac{1}{2}} e^{tA^{\frac{1}{2}}} \varphi|^2 + 1$ then, the above inequality implies

$$\frac{d}{dt} y(t) \le C_5 (y(t))^{\frac{7}{2}} \le 2^7 C_5 (y(0))^{\frac{7}{2}},$$

for some positive constant C_5 and for all $t \in [0, T^*)$.

Therefore we conclude that

$$y(t) \le y(0) + 2^7 C_5 (y(0))^{\frac{7}{2}} t \le y(0) + 2^7 C_5 (y(0))^{\frac{7}{2}} T^*.$$

Now we demand the right-hand side to be less than $2y(0)$. Then we find $T^* = 2^{-7} C_5^{-1} (|\varphi_0| + 1)^{-\frac{5}{2}}$. That is to say, $\varphi(t)$ belongs to G_t for $t \in [0, T^*]$ when φ_0 is in $\mathbb{H}$ and T^* is a lower bound for $\bar{T}$. Since we showed the global existence of the weak solution with uniform bound in $\mathcal{H}$ that depends on the initial data, we can repeat this procedure all the way to infinity. Indeed, suppose $|\varphi(t)| \le R_0$ and define $v(s) = \varphi(s + t)$ for $0 \le s, t \le T^*$, then $|v(0)| = |\varphi(t)| \le R_0$, for all $t \ge 0$. By the above argument, $v(s) = \varphi(s + t) \in G_s$, and therefore $v(T^*) = \varphi(T^* + t) \in G_{T^*}$, for $0 \le t \le T^*$. We continue this procedure to obtain the result $\varphi(t) \in G_{T^*}$ for all $t \ge T^*$. ☐

6. Inertial Manifold

In this section, we prove the existence of a finite-dimensional inertial manifold (IM) which contains the global attractor. An inertial manifold is a globally invariant Lipschitz manifold which attracts all the bounded sets at an exponential rate, justifying the use of classical tools for investigating the finite-dimensional behavior of dissipative PDEs.

Using the so-called Spectral Barrier approach [9] we will show that the system (5) has a finite-dimensional IM provided that a gap condition in the spectrum of its linear part is satisfied. There are other approaches for proving the existence of an inertial manifold, for instance the Lyapunov-Perron and Spectral Blocking [13] and [7] (see also [6], [8], and [14]). Other approaches have also been developed, such as the principle of spatial averaging method [29] and the classical graph transform technique due to Hadamard [30] (see also [22]). All of these methods require the spectral gap condition as a sufficient condition for the existence of an IM.

Following [9], we first recall the definition of a spectral barrier and the sufficient conditions required to guarantee the existence of an IM. Let us consider an equation of the form

$$\frac{d\boldsymbol{u}}{dt} + \Lambda \boldsymbol{u} + \mathbf{R}(\boldsymbol{u}) = 0, \tag{37a}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0. \tag{37b}$$

The operator Λ is self-adjoint and positive with domain $D(\Lambda) \subset \mathcal{H}$. The nonlinear map $\mathbf{R}$ is defined on $D(\Lambda)$ satisfying the conditions:

($R1$) There exists $r_0 > 0$ such that $\mathbf{R}(\boldsymbol{u}) = 0$ for all $\boldsymbol{u}$ satisfying $|\boldsymbol{u}| \ge r_0$.

($R2$) There exist $a_j, \lambda_{0j}, 0 < a_j \le 1,\ 0 < \lambda_{0j} < \infty$, for $1 \le j \le k$, such that for every $\boldsymbol{u}_1, \boldsymbol{u}_2$ in $D(\Lambda)$,

$$(\mathbf{R}(\boldsymbol{u}_1) - \mathbf{R}(\boldsymbol{u}_2), \boldsymbol{u}_1 - \boldsymbol{u}_2) \ge -\sum_{j=1}^{k} \lambda_{0j}^{a_j} |\boldsymbol{u}_1 - \boldsymbol{u}_2|^{2a_j} |\Lambda^{\frac{1}{2}} (\boldsymbol{u}_1 - \boldsymbol{u}_2)|^{2(1-a_j)}. \tag{38}$$

We also assume that the initial value problem (37) is solved by a semigroup $\{S(t)\}_{t\geq 0}$ of nonlinear injective operators such that

$$S(t)\colon \mathcal{H} \to D(\Lambda),$$

which is continuous for any $t > 0$.

Definition 3. [9] A number λ , $0 < \lambda < \infty$ is called a spectral barrier for equation (37) if for every distinct $\boldsymbol{u}_1, \boldsymbol{u}_2 \in D(\Lambda)$ satisfying

$$(\Lambda(\boldsymbol{u}_1 - \boldsymbol{u}_2), \boldsymbol{u}_1 - \boldsymbol{u}_2) = \lambda |\boldsymbol{u}_1 - \boldsymbol{u}_2|^2,$$

it follows that

$$|(\Lambda - \lambda I)(\boldsymbol{u}_1 - \boldsymbol{u}_2)|^2 + (\mathbf{R}(\boldsymbol{u}_1) - \mathbf{R}(\boldsymbol{u}_2), (\Lambda - \lambda I)(\boldsymbol{u}_1 - \boldsymbol{u}_2)) > 0.$$

Theorem 7. [9] *Let* λ *be a spectral barrier for (37) satisfying* $\lambda > \lambda_{0j}$, *for all* $j = 1, \ldots, k$, *and* $P_\lambda \mathcal{H}$ *is finite-dimensional, where* P_λ *is the* L^2 *orthogonal projection from* $\mathcal{H}$ *onto* $\mathcal{H}_\lambda = \text{span}\ \{e^{ij\theta}\colon \text{eigenvalues } \lambda_j \in [0, \lambda)\}$. *Assume that* λ *does not belong to the spectrum of* Λ. *Then there is a finite-dimensional inertial manifold for (37) which is represented as a graph of a Lipschitz function over* H_λ.

We now consider our system (5) and show the following.

Theorem 8. *There exists a finite-dimensional inertial manifold for the system of equations (5).*

Proof. Based on Theorem 1 and Corollary 2, system (5) has an absorbing ball in $L^\infty(\Omega)$. Indeed, since the weak solution becomes instantaneously a strong solution (see Remark 3), by the Sobolev imbedding theorem, we note that $\mathcal{V} \subset L^\infty(\Omega)$, and thus, by Agmon's inequality,

$$\limsup_{t\to\infty} \|\boldsymbol{u}(t)\|_{L^\infty(\Omega)} \leq \limsup_{t\to\infty} (|\boldsymbol{u}(t)|^{\frac{1}{2}}\ \|\boldsymbol{u}(t)\|^{\frac{1}{2}}) \leq (\rho_0 \rho_1)^{\frac{1}{4}} =: \rho_\infty.$$

Therefore, every bounded set in $\mathcal{H}$ is carried by the solution operator of (5) within a finite time into the set

$$2\mathcal{B} = \{\boldsymbol{u} \in L^\infty(\Omega) : \|\boldsymbol{u}\|_{L^\infty(\Omega)} \leq 2\rho_\infty\},$$

where $r\mathcal{B}$ denotes the set $r\mathcal{B} = \{\boldsymbol{u} \in L^\infty(\Omega) : \|\boldsymbol{u}\|_{L^\infty(\Omega)} \leq r\rho_\infty\}$.

We modify our system of equations outside the set $\mathcal{B}$ to obtain

$$\frac{d\boldsymbol{u}}{dt} + \tilde{\boldsymbol{L}}\boldsymbol{u} + \mathbf{R}(\boldsymbol{u}) = 0, \tag{39}$$

where

$$\tilde{\boldsymbol{L}}\boldsymbol{u} = \left(K^{-1}\frac{\nu}{2}A\varphi, 0, 0\right),$$

$$\tilde{\mathbf{f}}(\boldsymbol{u}) = \left(-aK^{-1}(\psi_c(\Phi + \varphi) - \overline{\psi_c}), -\frac{1}{l_c}(\overline{\psi_c} - \Psi), \frac{-1}{4l_cB^2}(\Phi - F_T^{-1}(\Psi))\right),$$

$$\begin{aligned}\mathbf{R}(\boldsymbol{u}) &= \left(\frac{1}{2}K^{-1}\frac{\partial\varphi}{\partial\theta}, 0, 0\right) + \chi(\boldsymbol{u})\tilde{\mathbf{f}}(\boldsymbol{u}) = (R_1(\boldsymbol{u}), R_2(\boldsymbol{u}), R_3(\boldsymbol{u})),\\ \chi(\boldsymbol{u}) &= \Theta\left(\frac{\|\boldsymbol{u}\|_{L^\infty(\Omega)}}{\rho_\infty}\right),\end{aligned}$$

and Θ is a smooth, nonincreasing function on $[0, \infty)$ such that

$$\Theta(r) = \begin{cases} 1 & \text{for } r \leq 1,\\ 0 & \text{for } r \geq 3,\end{cases}$$

with $|\Theta'(r)| \leq 1$ for all $r \in [0, \infty)$. Notice here we include the $\frac{1}{2}K^{-1}\frac{\partial\varphi}{\partial\theta}$ term in the nonlinear part of the equation. Therefore operator $\tilde{\boldsymbol{L}}$ is slightly different from (5). We also observe that $\mathbf{R}(\boldsymbol{u})$ does not satisfy condition (R1). However, one can easily check that in this case the proof and results of Theorem 7 remain valid. Since $\Phi(t)$ and $\Psi(t)$ depend on the time variable t only, to check the rest of the conditions of Theorem 7 it is enough to observe that the operator $K^{-1}A$ is self-adjoint and positive.

Furthermore, we notice that for $\boldsymbol{u} = (\varphi, \Phi, \Psi)$,

$$\begin{aligned}\|\boldsymbol{u}\| &= \|\varphi\| + |\Phi| + |\Psi| &\text{(40a)}\\ &\leq \|K\|^{\frac{1}{2}}_{\mathcal{L}(\mathbb{H}\times\mathbb{R})}\left|\left(K^{-1}\frac{\partial^2}{\partial\theta^2}\boldsymbol{u}, \boldsymbol{u}\right)\right|^{\frac{1}{2}} + |\Phi| + |\Psi| &\text{(40b)}\\ &\leq K_\nu^2|(\tilde{\mathbf{L}}\boldsymbol{u}, \boldsymbol{u})|^{\frac{1}{2}} + M_0|\boldsymbol{u}|, &\text{(40c)}\end{aligned}$$

for some positive constant M_0 and $K_\nu = (\sqrt{\frac{2}{\nu}}\|K\|^{\frac{1}{2}}_{\mathcal{L}(\mathbb{H}\times\mathbb{R})} + 1)^{1/2}$.

Let us now check condition (R2). Here we observe that since $\frac{\partial}{\partial\theta}$ is an antisymmetric operator, then for every $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathcal{V}$,

$$(\mathbf{R}(\boldsymbol{u}_1) - \mathbf{R}(\boldsymbol{u}_2), \boldsymbol{u}_1 - \boldsymbol{u}_2) = (\chi(\boldsymbol{u}_1)\,\tilde{\mathbf{f}}(\boldsymbol{u}_1) - \chi(\boldsymbol{u}_2)\tilde{\mathbf{f}}(\boldsymbol{u}_2), \boldsymbol{u}_1 - \boldsymbol{u}_2). \tag{41}$$

Let $\boldsymbol{u}_1, \boldsymbol{u}_2 \in D(A) \times \mathbb{R} \times \mathbb{R}$ be given. Then $\boldsymbol{u}_1, \boldsymbol{u}_2$ satisfy one of the following cases or their symmetric counter parts: Case 1: $\boldsymbol{u}_1, \boldsymbol{u}_2 \in 5\mathcal{B}$; Case 2: $\boldsymbol{u}_1 \in (3\mathcal{B})^c$, $\boldsymbol{u}_2 \in (5\mathcal{B})^c$; and Case 3: $\boldsymbol{u}_1 \in 3\mathcal{B}$, $\boldsymbol{u}_2 \in (5\mathcal{B})^c$. Here, we consider only Case 1. However, Case 2 and Case 3 can be proved similarly.

- Case 1: $\boldsymbol{u}_1, \boldsymbol{u}_2 \in 5\mathcal{B}$

For simplicity, we also denote $\overline{\psi_c^i} = \overline{\psi_c(\Phi_i + \varphi_i)}$ for $i = 1, 2$.

$$\begin{aligned}&(\mathbf{R}(\boldsymbol{u}_1) - \mathbf{R}(\boldsymbol{u}_2), \boldsymbol{u}_1 - \boldsymbol{u}_2)\\ &\quad = -(\chi(\boldsymbol{u}_1)(K^{-1}(\psi_c(\Phi_1 + \varphi_1) - \psi_c(\Phi_2 + \varphi_2)) - K^{-1}(\overline{\psi_c^1} - \overline{\psi_c^2})), \varphi_1 - \varphi_2)\\ &\quad - (\chi(\boldsymbol{u}_1)\,(\overline{\psi_c^1} - \overline{\psi_c^2}), \Phi_1 - \Phi_2) + (\chi(\boldsymbol{u}_1)(F_T^{-1}(\Psi_1) - F_T^{-1}(\Psi_2)), \Psi_1 - \Psi_2)\\ &\quad - ((\chi(\boldsymbol{u}_1) - \chi(\boldsymbol{u}_2))\,K^{-1}(\psi_c(\Phi_2 + \varphi_2) - \overline{\psi_c^2}), \varphi_1 - \varphi_2)\\ &\quad - ((\chi(\boldsymbol{u}_1) - \chi(\boldsymbol{u}_2))(\overline{\psi_c^2} - \Psi_2), \Phi_1 - \Phi_2)\\ &\quad - ((\chi(\boldsymbol{u}_1) - \chi(\boldsymbol{u}_2))(\Phi_2 - F_T^{-1}(\Psi_2)), \Psi_1 - \Psi_2),\end{aligned}$$

and by the mean-value theorem we have

$$
\begin{aligned}
&\geq -\|K^{-1}\|_{\mathcal{L}(\mathbb{H}\times\mathbb{R})} \,(\sup_{\boldsymbol{u}\in 5\mathcal{B}} |\psi_c'(\Phi+\varphi)| + \sup_{\boldsymbol{u}\in 5\mathcal{B}} |\overline{\psi_c}'(\Phi+\varphi)|) \\
&\qquad\qquad\qquad \times (|\varphi_1-\varphi_2|^2 + |\varphi_1-\varphi_2|\,|\Phi_1-\Phi_2|) \\
&\quad - \sup_{\boldsymbol{u}\in 5\mathcal{B}} |\overline{\psi_c}'(\Phi+\varphi)|(|\Phi_1-\Phi_2|^2 + |\varphi_1-\varphi_2|\,|\Phi_1-\Phi_2|) \\
&\quad - \sup_{\boldsymbol{u}\in 5\mathcal{B}} |(F_T^{-1})'(\Psi)|\,|\Psi_1-\Psi_2|^2 - \frac{\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_{L^\infty(\Omega)}}{\rho_\infty} \\
&\quad \times \max\left\{2\|K^{-1}\|_{\mathcal{L}(\mathbb{H}\times\mathbb{R})} \sup_{\boldsymbol{u}\in 5\mathcal{B}} |\psi_c(\Phi+\varphi)|,\ \sup_{\boldsymbol{u}\in 5\mathcal{B}} |\psi_c(\Phi+\varphi)| + 5\rho_\infty,\right. \\
&\qquad\qquad \left.(5\rho_\infty)^{\frac{1}{2}} + 5\rho_\infty\right\} C_0|\boldsymbol{u}_1-\boldsymbol{u}_2|.
\end{aligned}
$$

Since $\sup_{\boldsymbol{u}\in 5\mathcal{B}} |\psi_c'(\Phi+\varphi)| \leq \sup_{\{\|\varphi\|_{L^\infty}\leq 5\rho_\infty, \|\Phi\|_{L^\infty}\leq 5\rho_\infty\}} |\psi_c'(\Phi+\varphi)|$ and similarly for $\psi_c(\Phi+\varphi)$ and $(F_T^{-1})'(\Psi)$, we find

$$
\begin{aligned}
(\mathbf{R}(\boldsymbol{u}_1) - \mathbf{R}(\boldsymbol{u}_2), \boldsymbol{u}_1-\boldsymbol{u}_2) \;\geq\; & -M_1|\varphi_1-\varphi_2|^2 - M_2|\Phi_1-\Phi_2|^2 - M_3|\Psi_1-\Psi_2|^2 \\
& - M_4\frac{\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_{L^\infty(\Omega)}}{\rho_\infty}|\boldsymbol{u}_1-\boldsymbol{u}_2|,
\end{aligned}
$$

by using Agmon's inequality and (40),

$$
\begin{aligned}
&\geq\; -\max\{M_1, M_2, M_3\}\,|\boldsymbol{u}_1-\boldsymbol{u}_2|^2 \\
&\quad - \frac{M_4}{\rho_\infty}(K_\nu^2|\tilde{\mathbf{L}}^{\frac{1}{2}}(\boldsymbol{u}_1-\boldsymbol{u}_2)| + M_0|\boldsymbol{u}_1-\boldsymbol{u}_2|)^{\frac{1}{2}}|\boldsymbol{u}_1-\boldsymbol{u}_2|^{\frac{3}{2}} \\
&\geq\; -\max\left\{M_1, M_2, M_3, \frac{\sqrt{M_0}M_4}{\rho_\infty}\right\}|\boldsymbol{u}_1-\boldsymbol{u}_2|^2 - \frac{M_4}{\rho_\infty}K_\nu|\tilde{\mathbf{L}}^{\frac{1}{2}}(\boldsymbol{u}_1-\boldsymbol{u}_2)|^{\frac{1}{2}}\,|\boldsymbol{u}_1-\boldsymbol{u}_2|^{\frac{3}{2}}.
\end{aligned}
$$

Therefore, condition (R2) holds where $\lambda_{01} = \max\{M_1, M_2, M_3, \sqrt{M_0}M_4/\rho_\infty\}$ and $\lambda_{02} = (M_4K_\nu/\rho_\infty)^{4/3}$, $a_1 = 1$, and $a_2 = 3/4$.

Let $\boldsymbol{u}_1, \boldsymbol{u}_2 \in D(A)\times\mathbb{R}\times\mathbb{R}$ and let λ be given by $(\tilde{\mathbf{L}}(\boldsymbol{u}_1-\boldsymbol{u}_2), \boldsymbol{u}_1-\boldsymbol{u}_2) = \lambda|\boldsymbol{u}_1-\boldsymbol{u}_2|^2$. From Definition 3, λ is a spectral barrier for the system (39) provided that

$$
|(\tilde{\mathbf{L}}-\lambda I)(\boldsymbol{u}_1-\boldsymbol{u}_2)|^2 + (\mathbf{R}(\boldsymbol{u}_1) - \mathbf{R}_{\mathbf{2}}(\boldsymbol{u}_2), (\tilde{\mathbf{L}}-\lambda I)(\boldsymbol{u}_1-\boldsymbol{u}_2)) > 0. \tag{42}
$$

Notice again that since $\frac{\partial}{\partial\theta}$ is antisymmetric, (42) holds provided that

$$
|(\tilde{\mathbf{L}}-\lambda I)(\boldsymbol{u}_1-\boldsymbol{u}_2)|^2 > |\chi(\boldsymbol{u}_1)\tilde{\mathbf{f}}(\boldsymbol{u}_1) - \chi(\boldsymbol{u}_2)\tilde{\mathbf{f}}(\boldsymbol{u}_2)|\,|(\tilde{\mathbf{L}}-\lambda I)(\boldsymbol{u}_1-\boldsymbol{u}_2)|.
$$

We now find an upper bound for $|\chi(\boldsymbol{u}_1)\tilde{\mathbf{f}}(\boldsymbol{u}_1) - \chi(\boldsymbol{u}_2)\tilde{\mathbf{f}}(\boldsymbol{u}_2)|$. Here again we need to cover different cases. However, we only show Case 1, and the rest of the cases can be justified similarly.

• Case 1: $\boldsymbol{u}_1, \boldsymbol{u}_2 \in 5\mathcal{B}$

$$\begin{aligned} |\chi(\boldsymbol{u}_1)\tilde{\mathbf{f}}(\boldsymbol{u}_1) - \chi(\boldsymbol{u}_2)\tilde{\mathbf{f}}(\boldsymbol{u}_2)| &\le |\chi(\boldsymbol{u}_1)(\tilde{\mathbf{f}}(\boldsymbol{u}_1) - \tilde{\mathbf{f}}(\boldsymbol{u}_2))| + |(\chi(\boldsymbol{u}_1) - \chi(\boldsymbol{u}_2))\tilde{\mathbf{f}}(\boldsymbol{u}_2)| \\ &\le |\tilde{\mathbf{f}}(\boldsymbol{u}_1) - \tilde{\mathbf{f}}(\boldsymbol{u}_2)| + \frac{\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{L^\infty(\Omega)}}{\rho_\infty}|\tilde{\mathbf{f}}(\boldsymbol{u}_2)| \\ &\le \sup_{\boldsymbol{u}\in 5\mathcal{B}} |\tilde{\mathbf{f}}'(\boldsymbol{u})||\boldsymbol{u}_1 - \boldsymbol{u}_2| + \sup_{\boldsymbol{u}\in 5\mathcal{B}} |\tilde{\mathbf{f}}(\boldsymbol{u})|, \frac{1}{\rho_\infty}\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{L^\infty(\Omega)} \end{aligned}$$

by Agmon's inequality, (40c), and the definition of λ we have

$$\le \left(M_5 + \frac{M_6}{\rho_\infty} + \frac{M_7}{\rho_\infty} K_\nu |\lambda|^{\frac{1}{4}} \right) |\boldsymbol{u}_1 - \boldsymbol{u}_2|.$$

Let C_1, C_2 explicitly given constants that depend on $a, m, B, l_c, \rho_\infty$, and $\|K\|_{\mathcal{L}(\mathbb{H}\times\mathbb{R})}$; then (42) holds provided that

$$|(\tilde{\boldsymbol{L}} - \lambda I)(\boldsymbol{u}_1 - \boldsymbol{u}_2)| > (C_1 + C_2|\lambda|^{\frac{1}{4}})|\boldsymbol{u}_1 - \boldsymbol{u}_2|. \tag{43}$$

We denote the eigenvalues of the operator $\tilde{\mathbf{L}}$ by $\tilde{\lambda}_k = \frac{k^3}{k+am}$, $k = 1, 2, \ldots$. Let N be large enough to be determined later, and let us take $\lambda = \frac{\tilde{\lambda}_{N+1}+\tilde{\lambda}_N}{2}$. As a result we have

$$|(\tilde{\boldsymbol{L}} - \lambda I)(\boldsymbol{u}_1 - \boldsymbol{u}_2)| \ge \frac{\tilde{\lambda}_{N+1} - \tilde{\lambda}_N}{2}|\boldsymbol{u}_1 - \boldsymbol{u}_2|.$$

Hence, λ satisfies (43) if

$$\frac{\tilde{\lambda}_{N+1} - \tilde{\lambda}_N}{2} > C_1 + C_2 \left(\frac{\tilde{\lambda}_{N+1} + \tilde{\lambda}_N}{2} \right)^{\frac{1}{4}}.$$

Substituting $\tilde{\lambda}_N = \frac{N^3}{N+am}$, the above inequality then holds for every N large enough satisfying

$$N > C_3 + C_4 N^{\frac{1}{2}}, \tag{44}$$

where C_3, C_4 are positive constants which depend on a, m, l_c, ρ_∞, and $\|K\|_{\mathcal{L}(\mathbb{H}\times\mathbb{R})}$.

Finally, we need to check that

$$\lambda > \max\{\lambda_{01}, \lambda_{02}\}. \tag{45}$$

Since we have for $N \gg 1$

$$\lambda = \frac{2N^4 + (4 + 2am)N^3 + (3 + 3am)N^2 + (1 + 3am)N + am}{2(N + am)(N + am + 1)} \sim N^2,$$

it is clear that we can find $N \gg 1$ such that both (44) and (45) hold. Therefore, Theorem 7 guarantees the existence of a finite-dimensional inertial manifold of dimension N. $\square$

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