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Traveling Wave Solutions for a Class of One-Dimensional Nonlinear Shallow Water Wave Models

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In this paper we consider a class of one-dimensional nonlinear shallow water wave models that support weak solutions. We construct new traveling wave solutions for these models. Moreover, we show that these new traveling wave solutions are stable.

KEY WORDS: Traveling wave solutions; shallow water models.

1. INTRODUCTION

In this paper, we shall study traveling wave solutions for a set of one-dimensional nonlinear, nonlocal, evolutionary partial differential equations. This class of equations originally arose at quadratic order in the asymptotic expansion for shallow water waves [4, 10]. The famous Korteweg–de Vries equation – which is nonlinear, but local – arises uniquely at linear order in this shallow water wave expansion. At quadratic order, a broad class of asymptotically equivalent equations arises

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[10], including the class that we shall investigate. Remarkably, three-dimensional incompressible versions of these equations also arose in recent studies of the turbulence closure problem, obtained by averaging the exact fluid equations at constant Lagrangian coordinate, then making the Taylor hypothesis for frozen-in turbulence [5–8,12,13,15,16,18,21,22,25]. We shall restrict our present considerations to one-dimensional equations.

We shall consider the following Cauchy problem on the whole line, $\mathbb{R}$:

$$u_t + uu_x = -\tau_x, \tag{1}$$

$$\tau = \frac{1}{2}\int \left(\beta u^2(y) + \frac{\alpha^2}{2}u_y^2(y)\right) e^{-|x-y|/\alpha}\,\mathrm{d}y, \tag{2}$$

$$u(x,0) = u_0(x), \tag{3}$$

where $\alpha^2 \geqslant 0$ is a constant, $\beta \geqslant 0$ is a bifurcation parameter, and $u_0(x)$ is the initial condition.

Notice that

(i) For $\alpha^2 = \beta = 0$, the system (1)–(3) becomes the inviscid Burgers equation (cf. [3,20,23]).
(ii) For $\beta = 0$ and $\alpha^2 > 0$, the system (1)–(3) becomes the inviscid Burgers-Alpha equations (cf. [19]).
(iii) For $\beta = 1$ and $\alpha^2 > 0$, the system (1)–(3) becomes the inviscid one dimensional Camassa–Holm (CH) equation (cf. [4]).

Starting from the pioneering work of Burgers [3,23] and Hopf [20] the Burgers equation (especially, the viscous Burgers–Hopf equation) has always been used as a simple model to study shocks, turbulence and other nonlinear phenomena in fluids (see, for example, [11,24,26] and references therein). The system (1)–(3) is a nonlocal nonlinear deformation of the Burgers equation. However, the qualitative nature of the solutions to (1)–(3) are very different from those of the Burgers equation. First, the solutions for the Burgers equation blow up, in finite time, if the initial data has a negative slope. Specifically, the slope of any such solution becomes infinite in finite time and the function becomes discontinuous. In particular, the H^1 Sobolev norm, [1], of the solution blows up in finite time. On the other hand, the H^1 Sobolev norm of the solutions to (1)–(3) is conserved. In [27], the authors used this very fact to show the global existence of the weak solutions to the one dimensional CH equation, to which Eqs. (1)–(3) restrict for the case $\beta = 1$. (A weak solution u is defined as a solution that satisfies the equation in the distribution sense and which also belongs to $C([0,T]; H^1)$, for every $T > 0$.)

Second, unlike the Burgers equation, the CH equation that arises from Eqs. (1)–(3) for the case $\beta = 1$ admits special solutions, namely the peakons, (cf. [4])

$$u(x,t) = \sum_{i=1}^{N} P_i(t) e^{-|x - Q_i(t)|/\alpha}. \tag{4}$$

In this expression for the CH peakon solutions to (1)–(3) for the case $\beta = 1$, the parameters $P_i(t)$ and $Q_i(t)$ for $i = 1, 2, \dots, N$ satisfy the canonical Hamiltonian equations,

$$\mathrm{d}Q_i/\mathrm{d}t = \partial H/\partial P_i \text{ and } \mathrm{d}P_i/\mathrm{d}t = -\partial H/\partial Q_i, \quad i = 1, 2, \dots, N,$$

with Hamiltonian,

$$H = \frac{1}{2} \sum_{i,j=1}^{N} P_i(t) P_j(t) e^{-|Q_i(t) - Q_j(t)|/\alpha}.$$

The CH peakons (4) for the case $\beta = 1$ in Eqs. (1)–(3) are soliton solutions. Consequently, they dominate the corresponding initial value problem (cf. [4, 14]). The CH initial value problem is solved via the inverse scattering transformation (IST) method using the isospectral eigenvalue problem for the CH equation that was discovered in [4]. For any initially confined distribution of fluid velocity $u_0(x)$, the CH isospectral problem was shown in [4] to possess purely discrete spectrum with eigenvalues $c_i, i = 1, 2, \dots$. These eigenvalues correspond to the asymptotic speeds of the peakons, $c_i = \lim_{t\to\infty} P_i(t)$. For more discussion of the CH initial value problem, see [2, 4, 14]. In [9], the authors also proved that the single CH peakon traveling wave solution for $\beta = 1$,

$$u(x,t) = c e^{-|x - ct|/\alpha}, \tag{5}$$

is stable. As a result of this stability, one can conclude that the single peakon solution is unique. However, regardless of the special peakon solutions (4) that exist for finite N, the question of uniqueness for weak solutions to the one-dimensional CH equation as a partial differential equation is still open [27].

In this article we shall consider the system (1)–(3) which is similar to the CH equation. Indeed, one can follow the proofs of [27] to show the existence of the weak solutions, i.e., solutions u that satisfy the equation in the distribution sense and which also belong to $C([0, T]; H^1)$, for every $T > 0$. Here, we shall show the existence of traveling wave solutions for the system (1)–(3) for every $\alpha^2 > 0$ and $0 \leqslant \beta \leqslant 1$. As an immediate corollary of [9], one can show that this traveling wave solution is also stable.

Without loss of generality, we shall assume in what follows that $\alpha^2 = 1$ and $0 \leqslant \beta < 1$. The limit case $\beta = 1$ recovers the single peakon (5).

2. TRAVELING WAVE SOLUTION

Before we state our main Theorem, we need the following basic technical Lemma.

Lemma 2.1. *Let* $\{\xi_k\}_{k=1}^{\infty}$ *be a sequence of positive real number defined by induction:*

$$\xi_1 = 1, \tag{6}$$

$$\xi_k = \sum_{i+j=k;i,j\geqslant 1} \left(\frac{1}{2} - \frac{\beta + ij/2}{k^2-1}\right) \xi_i \xi_j \quad \text{for } k \geqslant 2. \tag{7}$$

Denote by

$$f(x) = \sum_{k=1}^{\infty} \xi_k x^k.$$

Then, there is $x^* > 0$*, such that* f *is well defined on* $|x| < x^*$*. Moreover,* $f(x)$ *has a continuous extension such that* $f(x^*) = 1$*, or* $\lim_{x\to x^{*-}} f(x) = 1$*.*

Proof. Let $\{\eta_k\}_{k=1}^{\infty}$ be a sequence of positive real number defined by induction:

$$\eta_1 = 1,$$
$$\eta_k = \sum_{i+j=k;i,j\geqslant 1} \frac{1}{2}\eta_i \eta_j \quad \text{for } k \geqslant 2.$$

Denote by

$$g(x) = \sum_{k=1}^{\infty} \eta_k x^k.$$

It is clear that $\xi_k \leqslant \eta_k$, for $k = 1, 2, \dots$. By the definition we have

$$g(x) = 1 - (1-2x)^{1/2}.$$

In other words g is well defined on $(-1/2, 1/2)$. Therefore, f is well defined, at least, on $(-1/2, 1/2)$. Let x^* be the radius of the convergence for f. Next, let us show that $f(x)$ has a continuous extension such that $f(x^*) = 1$. Namely, $\lim_{x\to(x^*)^-} f(x) = 1$. Denote by

$$h(x) = \sum_{k=2}^{\infty} \left(\sum_{i+j=k;i,j\geqslant 1} \frac{\beta + ij/2}{k^2-1} \xi_i \xi_j \right) x^k.$$

Therefore,

$$f - x = \frac{1}{2} f^2 - h.$$

By simple calculation we have, for $|x| < x^*$,

$$h(x) = \frac{\beta x}{2} \int_0^x \frac{f^2}{y^2} \mathrm{d}y + \frac{x}{4} \int_0^x (f')^2 \mathrm{d}y - \frac{\beta}{2x} \int_0^x f^2 \mathrm{d}y - \frac{1}{4x} \int_0^x y^2 (f')^2 \mathrm{d}y.$$

Moreover, we obtain

$$f'' + \frac{xf' - f}{x^2} = \frac{ff'}{x} - \left(\frac{1}{2} + \beta\right) \frac{f^2}{x^2} + \frac{1}{2} f'^2 + ff''. \tag{8}$$

Since our purpose is to prove $\lim_{x \to (x^*)^-} f(x) = 1$, let us assume that $x > 0$. By changing variable $x = e^z$, we have

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{1}{x} \frac{\mathrm{d}f}{\mathrm{d}z} \quad \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = \frac{1}{x^2} \frac{\mathrm{d}^2 f}{\mathrm{d}z^2} - \frac{1}{x^2} \frac{\mathrm{d}f}{\mathrm{d}z}.$$

As a result, Eq. (8) can be rewritten to

$$\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} - f = f \frac{\mathrm{d}^2 f}{\mathrm{d}z^2} + \frac{1}{2} \left(\frac{\mathrm{d}f}{\mathrm{d}z}\right)^2 - \left(\frac{1}{2} + \beta\right) f^2.$$

Notice that the above ODE is explicit independent of z. Thus, by assuming that $\mathrm{d}f/\mathrm{d}z = F(f)$, as an ansatz, we reach

$$(1 - f) F(f) \frac{\mathrm{d}F}{\mathrm{d}f} - \frac{1}{2} F^2(f) = f - \left(\frac{1}{2} + \beta\right) f^2.$$

Solving the above first-order ODE to obtain

$$\frac{\mathrm{d}f}{\mathrm{d}z} = F(f) = \left(\frac{1 - \lambda^2 f}{1 - f}\right)^{1/2} f,$$

where we define $\lambda = \left(\frac{1+2\beta}{3}\right)^{1/2}$. Therefore,

$$\left(\frac{1 - f}{1 - \lambda^2 f}\right)^{1/2} \frac{\mathrm{d}f}{f} = \mathrm{d}z = \frac{\mathrm{d}x}{x}.$$

By changing variable $f = (1 - z^2)/(1 - \lambda^2 z^2)$, we have

$$\frac{2(\lambda^2 - 1) z^2 \mathrm{d}z}{(1 - z^2)(1 - \lambda^2 z^2)} = \frac{\mathrm{d}x}{x}.$$

Solving the above first-order ODE to obtain an implicit analytic formula for function f:

$$\frac{4(1-\lambda)^{\frac{1}{\lambda}-1}}{(1+\lambda)^{\frac{1}{\lambda}+1}}\left(\frac{(1-\lambda^2 f)^{1/2}+\lambda(1-f)^{1/2}}{(1-\lambda^2 f)^{1/2}-\lambda(1-f)^{1/2}}\right)^{1/\lambda}\frac{(1-\lambda^2 f)^{1/2}-(1-f)^{1/2}}{(1-\lambda^2 f)^{1/2}+(1-f)^{1/2}}=x,$$

for $x>0$. Denote by

$$G(y)=\frac{4(1-\lambda)^{\frac{1}{\lambda}-1}}{(1+\lambda)^{\frac{1}{\lambda}+1}}\left(\frac{(1-\lambda^2 y)^{1/2}+\lambda(1-y)^{1/2}}{(1-\lambda^2 y)^{1/2}-\lambda(1-y)^{1/2}}\right)^{1/\lambda}$$
$$\times\frac{(1-\lambda^2 y)^{1/2}-(1-y)^{1/2}}{(1-\lambda^2 y)^{1/2}+(1-y)^{1/2}}.$$

By direct calculation we have

$$G'(y)=(1-\lambda^2 y)^2\left(\frac{1-y}{1-\lambda^2 y}\right)^{1/2}\frac{G(y)}{y}>0 \quad \text{for } 0<y<1. \tag{9}$$

By the Implicit Function Theorem, we conclude that f is well defined in

$$|x|<x^*=\frac{4(1-\lambda)^{\frac{1}{\lambda}-1}}{(1+\lambda)^{\frac{1}{\lambda}+1}} \tag{10}$$

and

$$\lim_{x\to x^{*-}} f(x)=1.$$

Since $\xi_k\geqslant 0, k=1,2,\ldots,$ one can easy to check that

$$\sum_{k=1}^{\infty}\xi_k(x^*)^k=f(x^*)=1. \qquad \square$$

Theorem 2.2. *For every $q>0$, denote by*

$$a_k=(x^*)^k\xi_k q \quad \textit{for } k\geqslant 1, \tag{11}$$

where ξ_k and x^* are as in (6), (7) and (10), respectively. Let

$$u_s(x,t)=p(x-qt)$$

with

$$p(x)=\sum_{k=1}^{\infty}a_k e^{-k|x|}.$$

Then u is a solution of the system (1)–(3) in the weak sense.

Proof. We need to check that u_s satisfies Eqs. (1) and (2). Notice that

$$(u_s)_t = -qp' \quad \text{and} \quad (u_s)_x = p'.$$

As a result, p should be a solution of the following equations in order that u_s satisfies Eqs. (1) and (2)

$$-qp' + (p^2/2 + \tau) = 0, \tag{12}$$

$$\tau - \tau'' = \beta p^2 + \frac{1}{2}(p')^2. \tag{13}$$

Notice that in case that

$$p(x) = \sum_{k=1}^{\infty} a_k e^{-k|x|},$$

we get

$$\tau(x) = \sum_{k=1}^{\infty} \tau_k e^{-k|x|},$$

with

$$\tau_1 = \sum_{k=2}^{\infty} \left(\frac{k}{k^2-1} \sum_{i+j=k} (\beta + ij/2) a_i a_j \right)$$

$$\tau_k = -\frac{k}{k^2-1} \sum_{i+j=k} \left(\beta + \frac{ij}{2} \right) a_i a_j \quad \text{for } k = 2, 3, \ldots .$$

As a result (12) and (13) are equivalent to $\{a_k\}_{k=1}^{\infty}$ satisfy

$$-qa_k + \sum_{i+j=k} \left(\frac{1}{2} - \frac{\beta + ij/2}{k^2-1} \right) a_i a_j = 0, \quad \text{for} \quad k \geqslant 2,$$

$$-qa_1 + \sum_{k=2}^{\infty} \left(\frac{k}{k^2-1} \sum_{i+j=k} (\beta + ij/2) a_i a_j \right) = 0,$$

By (11) and definition of $\{\xi_k\}_{k=1}^{\infty}$, we conclude that the above are true if

$$\sum_{k=2}^{\infty} \left(\frac{k}{k^2-1} \sum_{i+j=k} (\beta + ij/2) \xi_i \xi_j \right) x^{*k-1} = 1.$$

Denote by

$$\Phi(x) = \sum_{k=2}^{\infty} \left(\frac{k}{k^2-1} \sum_{i+j=k} (\beta + ij/2)\xi_i \xi_j \right) x^{k-1}.$$

It is clear that $\Phi(x) = h'(x)$, for $|x| < x^*$. By (8), we have $h'(x) = 1 + (1 - f(x))f'(x)$, for $|x| < x^*$. By (9), we have

$$\lim_{x \to x^{*-}} (1 - f(x))f'(x) = 0.$$

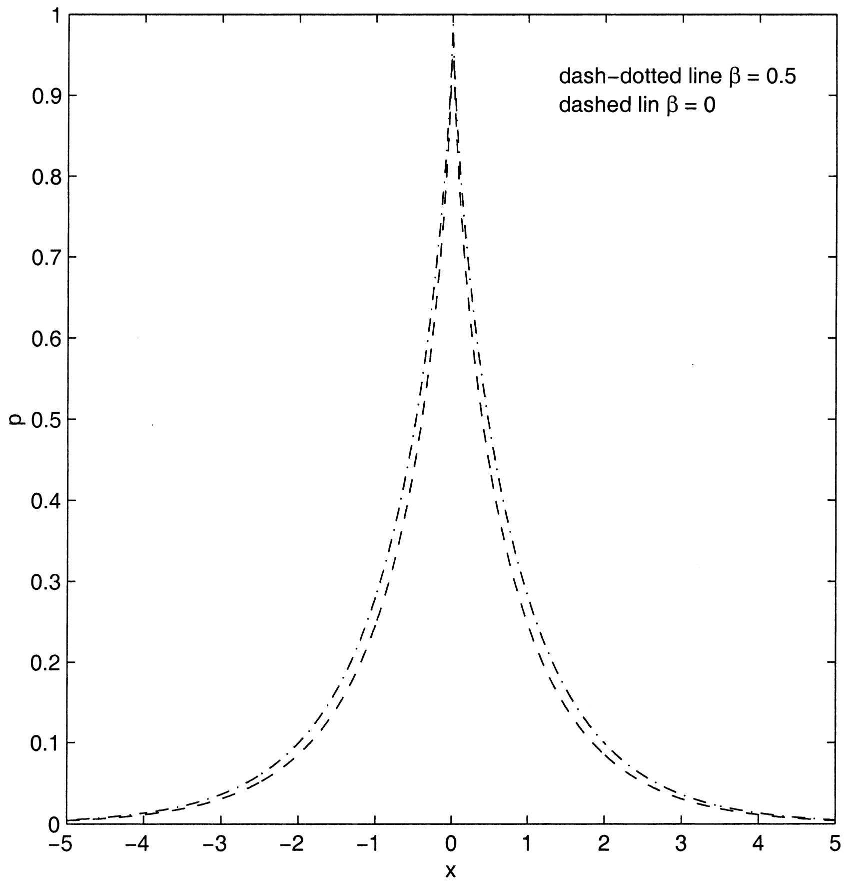

Figure 1. The traveling wave profile $p(x)$ for $\beta = 1/2$ and $\beta = 0$.

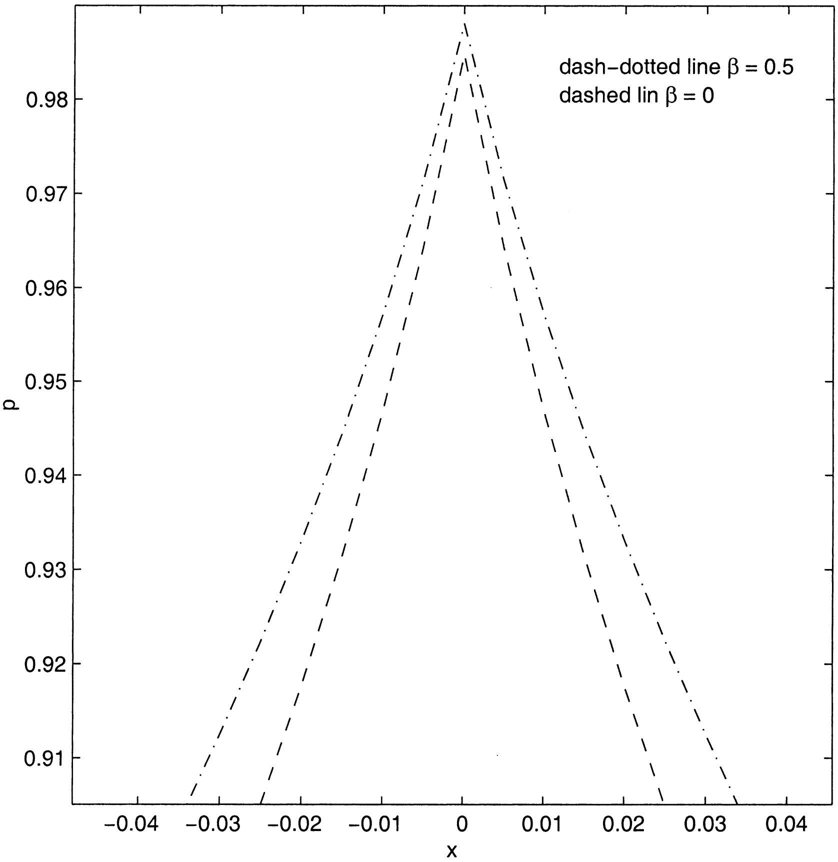

Figure 2. The zoom in profile for the traveling wave $p(x)$ near $x=0$ for $\beta=1/2$ and $\beta=0$.

Therefore,

$$\lim_{x\to x^{*-}} \Phi(x) = \lim_{x\to x^{*-}} h'(x) = 1.$$

Since $\xi_k \geqslant 0, k=1,2,\ldots,$ one can easy to check that

$$\sum_{k=2}^{\infty}\left(\frac{k}{k^2-1}\sum_{i+j=k}(\beta+ij/2)\xi_i\xi_j\right)x^{k-1} = \Phi(x^*) = 1.$$

This completes the proof. □

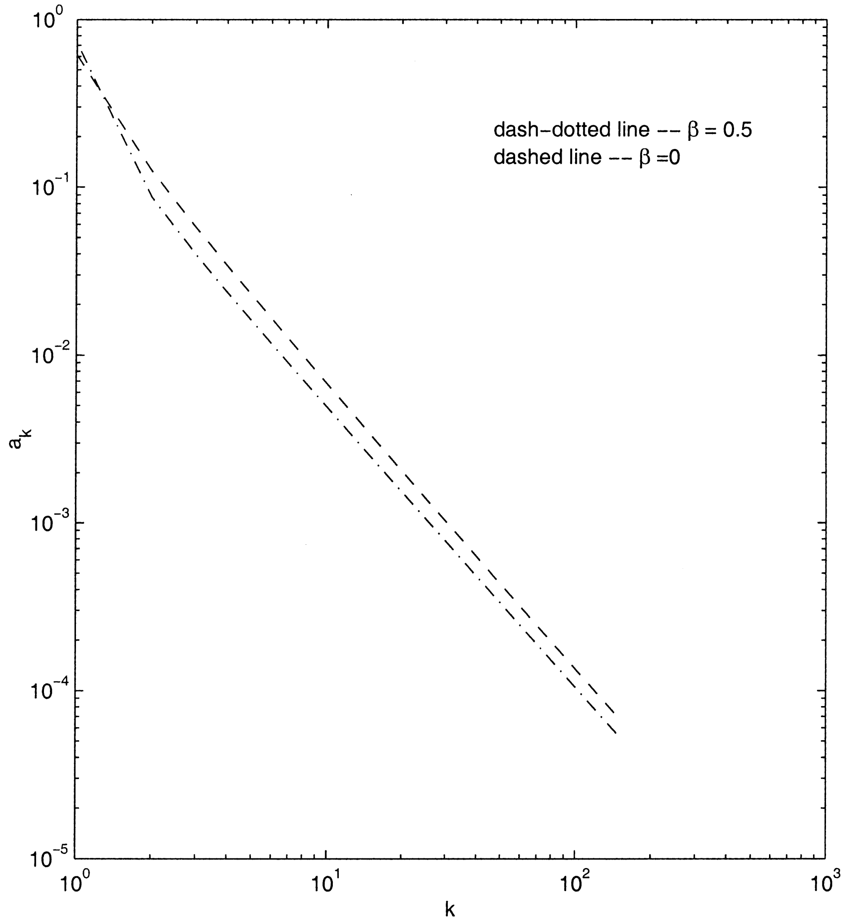

Figure 3. The coefficients a_k in the traveling wave $p(x)$ for $\beta = 1/2$ and $\beta = 0$.

Corollary 2.3. *The solution in Theorem 2.2 is stable. Namely, let* $\epsilon > 0$ *be given, and let* $u \in C([0, T]; H^1(\mathbb{R}))$ *be a solution to the system (1)–(3) with initial data* u_0 *satisfying*

$$\|u_0 - qp\|_{H^1} < C\epsilon^4,$$

then,

$$\|u(t) - qu_s\|_{H^1} < \epsilon.$$

Here C is a positive constant.

Proof. First, notice that, under the solutions of (1)–(3)

$$E(u) := \int \left(u^2 + u_x^2\right) \mathrm{d}x \quad \text{and} \quad F(u) := \int \left(\frac{4}{3}u^3 + uu_x\right) \mathrm{d}x$$

are conserved. Then one follows the techniques developed in [9] to complete the proof. □

In the Figures 1 and 2 we plot some of the profiles of the traveling wave solutions to illustrate the pronounced nonsmooth peak in the solution. Figure 3 indicates the algebraic decay rate of the coefficients a_k.

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REFERENCES

1. Adams, R. A. (1975). *Sobolev Spaces*, Academic Press, New York.
2. Beals, R., Sattinger, D. H., and Szmigielski, J. (1998). Acoustic scattering and the extended Korteweg-de Vries hierarchy. *Adv. Math.* **140**, 190–206.
3. Burgers, J. M. (1974). *Nonlinear Diffusion Equation*, Reidel, Dordrecht, Netherlands.
4. Camassa, R., and Holm, D. D. (1993). An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett.* **71**, 1661–1664.
5. Chen, S. Y., Foias, C., Holm, D. D., Olson, E. J., Titi, E. S., and Wynne, S. (1998). The Camassa–Holm equations as a closure model for turbulent channel flow. *Phys. Rev. Lett.* **81**, 5338.
6. Chen, S. Y., Foias, C., Holm, D. D., Olson, E. J., Titi, E. S., and Wynne, S. (1999). A connection between Camassa–Holm equations and turbulent flows in channels and pipes. *Phys. Fluids* **11**, 2343.
7. Chen, S. Y., Foias, C., Holm, D. D., Olson, E. J., Titi, E. S., and Wynne, S. (1999). The Camassa–Holm equations and turbulence. *Physica D* **133**, 49.
8. Chen, S. Y., Holm, D. D., Margolin, L. G., and Zhang, R. (1999). Direct numerical simulations of the Navier–Stokes alpha model. *Physica D* **133**, 66.
9. Constantin, A., and Strauss, W. A. (2000). Stability of Peakons. *Commun. Pure Appl. Math.* **53**, 603–610.
10. Dullin, H. R., Gottwald, G., and Holm, D. D. (2001). An integrable shallow water equation with linear and nonlinear dispersion. *Phys. Rev. Lett.* **87**, 4501–4504.
11. E, W., Khanin, K., Mazel, A., and Sinai, Y. (1997). Probability distribution functions for the random forced Burgers equation. *Phys. Rev. Lett.* **78**, 1904–1907.

12. Foias, C., Holm, D. D., and Titi, E. S. (2001). The Navier–Stokes-alpha model of fluid turbulence. *Physica D* **152**, 505.
13. Foias, C., Holm, D. D., and Titi, E. S. (2002). The three dimensional viscous Camassa–Holm equations, and their relation to the Navier–Stokes equations and turbulence theory. *J. Dyn. Diff. Eqns* **14**(2002), 1–35.
14. Fringer, O. B., and Holm, D. D. (2001). Integrable vs. nonintegrable geodesic soliton behavior. *Physica D* **150**, 237–263.
15. Geurts, B. G., and Holm, D. D. (2002). Alpha-modeling strategy for LES of turbulent mixing. In Drikakis, D., and Geurts, B. G. (eds.), *Turbulent Flow Computation*, Kluwer: London, pp. 237–278.
16. Holm, D. D. (1999). Fluctuation effects on 3D Lagrangian mean and Eulerian mean fluid motion. *Physica D* **133**, 215.
17. Holm, D. D., Marsden, J. E., and Ratiu, T. S. (1998). Euler–Poincaré equations and semidirect products with applications to continuum theories. *Adv. in Math.* **137**, 1.
18. Holm, D. D., Marsden, J. E., and Ratiu, T. S. (1998). Euler-Poincaré models of ideal fluids with nonlinear dispersion. *Phys. Rev. Lett*. **80**, 4173.
19. Holm, D. D., and Staley, M. F. (2003). Wave structures and nonlinear balances in a family of evolutionary PDEs. *SIAM J. Dyn. Sys.*, to appear.
20. Hopf, E. (1950). The partial differential equation $u_t + uu_x = u_{xx}$. *Comm. Pure Appl. Math.* **3**, 201–230.
21. Marsden, J. E., Ratiu, T. S., and Shkoller, S. (2000). The geometry and analysis of the averaged Euler equations and a new diffeomorphism group. *Geom. Funct. Anal.* **10**, 582.
22. Marsden, J. E., and Shkoller, S. (2001). Global well-posedness forthe Lagrangian averaged Navier–Stokes (LANS-α) equations on bounded domains. *Phil. Trans. R. Soc. Lond. A* **359**, 1449.
23. Nieuwstadt, F. T. M., and Steketee, J. A. (1994). *Selected Papers of J.M. Burgers*. Kluwer.
24. She, Z., Aurell, E., and Frisch, U. (1992). The inviscid Burgers equation with initial data of Brownian type. *Commun. Math. Phys.* **148**, 623–641.
25. Shkoller, S. (1998). Geometry and curvature of diffeomorphism groups with H^1 metric and mean hydrodynamics. *J. Func. Anal.* **160**, 337.
26. Sinai, Ya. G. (1992). Statistics of shocks in solutions of inviscid Burgers equation. *Commun. Math. Phys.* **148**, 601–621.
27. Xin, Z., and Zhang, P. (2000). On the weak solutions to a shallow water equation. *Commun. Pure Appl. Math.* **53**, 1411–1433.