On The Riemann Hypothesis For The Characteristic p Zeta Function

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1. Introduction

Let $\mathbf{F}_q$ be a finite field of q elements with characteristic p. Let A be the polynomial ring $\mathbf{F}_q[t]$. Let K be the completion of the quotient field $\mathbf{F}_q(t)$ of A at $t=0$. Thus, the field K is just the field of Laurent series $\mathbf{F}_q((t))$ with the discrete valuation determined by $v(t)=1$. Let Ω be the completion of an algebraic closure of K and let $\mathbf{Z}_p$ be the ring of p-adic integers. For $s=(x,y)\in\Omega^*\times\mathbf{Z}_p$ and a polynomial f of the form $1+tg(t)$ with $g(t)\in\mathbf{F}_q[t]$, one defines the exponential

$$f^s = x^{\deg(f)} f^y. \tag{1.1}$$

This is a well defined element in Ω. A polynomial f is said to be monic if $f(t)$ is of the form $1+tg(t)$. The constant 1 is the only monic polynomial of degree zero. By convention, the constant 1 is not an irreducible polynomial. The characteristic p zeta function in the theory of Drinfeld modules is then

$$\begin{aligned}
\zeta(A,s) &= \frac{1}{1-1/x}\prod_{\text{monic irred. } P\in A}\frac{1}{1-P^{-s}}\\
&= \frac{1}{1-1/x}\sum_{\text{monic } f\in A}\frac{1}{f^s}\\
&= \frac{1}{1-1/x}\sum_{d=0}^{\infty}(\frac{1}{x})^d\sum_{\text{monic } f\in A,\ \deg(f)=d}\frac{1}{f^y}\\
&= \sum_{d=0}^{\infty}(\frac{1}{x})^d\sum_{\text{monic } f\in A,\ 0\le\deg(f)\le d}\frac{1}{f^y}.
\end{aligned}\tag{1.2}$$

This definition of $\zeta(A,s)$ differs slightly from Goss' original definition in the sense that our uniformizer t corresponds to the uniformizer $1/T$ in Goss' definition. Note that the factor $1/(1-1/x)$ corresponds to the missing prime t in the above Euler product (or the missing monic prime 1 if one thinks of 1 as a prime).

The function $\zeta(A,s)$ is a characteristic p analogue over $\mathbf{F}_q[t]$ of the classical Riemann zeta function over $\mathbf{Z}$. It has many interesting properties shared by the Riemann zeta function. Special values of $\zeta(A,s)$ were first investigated by Carlitz [4]. In recent years, the function $\zeta(A,s)$ has been studied extensively in the arithmetic theory of Drinfeld modules, see Goss [7] for a comprehensive survey. For example, Goss [6] showed that for each fixed $y\in\mathbf{Z}_p$, the function $\zeta_y(A,x)=\zeta(A,(x,-y))$ is a t-adic entire function in $1/x$. Furthermore, the function $\zeta_y(A,x)$ is a polynomial in $1/x$ if y is a positive integer. Goss [6] also showed that $\zeta(A,s)$ has a trivial simple zero at negative A-even integers $s=(t^i,-i)$

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(where i is a positive integer divisible by $(q-1)$) and $\zeta(A,s)$ has no zeroes at negative A-odd integers $s=(t^i,-i)$ (where i is a positive integer not divisible by $(q-1)$). These results are consistent with what one knows about the Riemann zeta function. Further interesting analogues are Anderson-Thakur's formula [3] for the special values of $\zeta(A,s)$ in terms of the logarithm associated to higher Carlitz modules and Yu's [12] transcendental results on the special values of $\zeta(A,s)$.

In this paper, we study an analogue of the Riemann hypothesis for the characteristic p zeta function. The naive question would be to determine the location of the non-trivial zeroes of $\zeta_y(A,x)$ for each y. For classical congruence zeta functions over finite fields, the Riemann hypothesis is to determine the complex absolute values of the zeroes and poles (Weil-Deligne theorem) and to show that the absolute values of the zeroes and poles are of the form q^α, where $\alpha\in\frac{1}{2}\mathbf{Z}$. In our present case, the "complex" absolute value (the absolute value at "∞") is the t-adic absolute value. Thus, a strong form of the Riemann hypothesis for $\zeta(A,s)$ is to determine the t-adic absolute values of the zeroes of $\zeta(A,s)$, namely, the Newton polygon of the entire function $\zeta_y(A,x)$ for each y. This strong form of the Riemann hypothesis is our main concern here.

Our main result is an asymptotic formula for the coefficient of $(1/x)^d$ in $\zeta_y(A,x)$ described in terms of the q-digits of y. The main idea of the proof is to reduce the asymptotic formula to a family of minimizing problems in integer linear programming and the main difficulty is to solve the family of integer programming problems. As a consequence, we obtain a lower bound $P(y)$ for the Newton polygon of the entire function $\zeta_y(A,x)$ in terms of the q-digits of y. Our asymptotic formula tells exactly when the Newton polygon coincides with its lower bound $P(y)$. For example, if the digits in the q-expansion of y are either smaller than p or equal $(q-1)$, then the lower bound $P(y)$ coincides with the Newton polygon of $\zeta_y(A,x)$. In particular, the lower bound $P(y)$ coincides identically with the Newton polygon of $\zeta_y(A,x)$ when A is $\mathbf{F}_p[t]$. In the case that the Newton polygon of $\zeta_y(A,x)$ coincides with its lower bound $P(y)$, we also obtain an asymptotic formula for the roots of $\zeta_y(A,x)$.

For a general global L-series of Drinfeld modules and t-motives over the global ring $A=\mathbf{F}_q[t]$ (whose meromorphic continuation was conjectured by Goss [7] and established very recently by Taguchi and Wan [9] using Dwork's p-adic methods), it is too general and unreasonable to ask for a complete determination of the Newton polygons. However, one knows that for a fixed y, the t-adic absolute values of the zeroes and poles of a global L-series are always of the form q^α, where $\alpha\in\mathbf{Q}$. One form of the Riemann hypothesis for general global L-series is then the statement that for a fixed y, these rational exponents α have bounded denominators depending only on the L-series and y. One can strengthen this statement by requiring that the denominators be uniformly bounded in y. For the classical congruence zeta functions which are rational functions, the Weil-Deligne theorem shows that the denominator is at most 2. Even for these congruence zeta functions, the p-adic version of the Riemann hypothesis can already have denominators greater than 2, see for instance Wan [11].

A related but different version of the Riemann hypothesis for $\zeta(A,s)$ was first formulated by D. Goss. Denote the Riemann zeta function by $\zeta(\mathbf{Z},s)$ and define

$$\eta(s)=s(1-s)\Gamma(s/2)\pi^{-s/2}\zeta(\mathbf{Z},s).$$

It is well known that $\eta(s)$ is an entire function satisfying the functional equation $\eta(s)=\eta(1-s)$. The Riemann Hypothesis says that all zeroes of $\eta(s)$ are of the form $\frac{1}{2}+i\beta$.

Define

$$\theta(s) = \eta(i(s+\frac{1}{2})).$$

One checks that $\theta(s)$ is an entire power series in s with coefficients in the real numbers $\mathbf{R}$. In terms of $\theta(s)$, the Riemann Hypothesis becomes the simple statement that all zeroes of $\theta(s)$ are in $\mathbf{R}$, the completion of $\mathbf{Q}$ at infinity. A possible version of the Riemann hypothesis for the characterisitc p zeta function $\zeta(A, s)$ would then be that all zeroes of $\zeta_y(A, x)$ are in K, the completion of $\mathbf{F}_q(t)$ at the infinity $t = 0$. The first evidence is that the trivial zeroes of $\zeta(A, s)$ in (1.2) are trivially in K. For a general Dedekind domain A, Goss has constructed examples which show that the zeroes may be in an extension field (possibly inseparable) of K. Thus, Goss' version of the Riemann hypothesis for $\zeta(A, s)$ is roughly about the statement that the zeroes of $\zeta_y(A, x)$ are in a finite normal extension of K. Our lower bound $P(y)$ for the Newton polygon of $\zeta_y(A, x)$ has the property (see Lemma 2.1) that the horizontal projection of each side is of length 1. Thus, whenever the Newton polygon of $\zeta_y(A, x)$ coincides with the lower bound $P(y)$, the zeroes of $\zeta_y(A, x)$ are in K and are simple zeroes. In particular, Goss' version of the Riemann hypothesis for $\zeta(A, s)$ is true when $A = \mathbf{F}_p[t]$.

It should be possible to refine our method to show that for a unique factorization Dedekind domain A over $\mathbf{F}_q$ with a rational point at ∞, there is a positive integer d depending only on A such that for each $y \in \mathbf{Z}_p$, all zeroes (except for the first d zeroes) of $\zeta(A, (x, y))$ are in K, the completion of the quotient field of A at ∞. A new idea seems to be needed to treat the general case when A has class number greater than 1.

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2. Main Theorem

This section describes the asymptotic formula for the coefficients of $\zeta_y(A, x)$ and some of its consequences. By definition of $\zeta_y(A, x)$, one computes that

$$\begin{aligned}
\zeta_y(A, x) &= \zeta(A, (x, -y)) \\
&= 1 + \sum_{d=1}^{\infty} (\frac{1}{x})^d \sum_{\text{monic } f\in A,\ 0\le \deg(f)\le d} f^y \\
&= 1 + \sum_{d=1}^{\infty} (\frac{1}{x})^d \sum_{a_1,\cdots,a_d\in\mathbf{F}_q} (1 + a_1 t + a_2 t^2 + \cdots + a_d t^d)^y \\
&= 1 + \sum_{d=1}^{\infty} (\frac{1}{x})^d \sum_{h=0}^{\infty} \binom{y}{h} C_d(h), \qquad (2.1)
\end{aligned}$$

where

$$C_d(h) = \sum_{a_1,\cdots,a_d\in\mathbf{F}_q} (a_1 t + a_2 t^2 + \cdots + a_d t^d)^h, \ C_d(0) = 0. \qquad (2.2)$$

Let $v_d(y)$ be the t-adic valuation of the coefficient of $(1/x)^d$ in (2.1). The Newton polygon of $\zeta_y(A, x)$ is then the convex hull in the plane $\mathbf{R}^2$ of the lattice points $(d, v_d(y))$ $(d \ge 0)$.

The t-adic valuations of the zeroes of $\zeta_y(A, x)$ are completely described by its Newton polygon. Namely, if the Newton polygon has a side of slope s whose horizontal projection is of length ℓ, then $\zeta_y(A, x)$ has exactly ℓ zeroes with valuation s.

The Newton polygon is in general not easy to determine since the computation of the integers $v_d(y)$ can be difficult. It is, however, possible to give a good lower bound for the Newton polygon, namely a sequence of integers $w_d(y)$ with $w_d(y) \leq v_d(y)$ for all d. To describe our lower bound $w_d(y)$, we make use of the q-expansion of the p-adic integer $y \in \mathbf{Z}_p$:

$$y = y_0 + y_1 q + y_2 q^2 + \cdots, \ 0 \leq y_j \leq q-1. \tag{2.3}$$

Given a p-adic integer y in (2.3), We use its q-digits y_j to define four functions on the set of non-negative integers d. These four functions are the subscript function $j_d(y)$, the star function $y_{j_d}^*$, the truncation function $[y]_d$ and the weight function $w_d(y)$.

The subscript function $j_d(y)$ is the smallest non-negative integer j_d (if it exists) such that

$$d(q-1) \leq y_0 + y_1 + \cdots + y_{j_d}. \tag{2.4}$$

If such an integer j_d does not exist (this can occur only if y is a non-negative integer and thus there are only finitely many non-zero digits in (2.3)), then we skip the definition of j_d, $y_{j_d}^*$, $[y]_d$ and define $w_d(y) = \infty$. Otherwise, there is a unique non-negative integer $y_{j_d}^*$ (the star function) such that

$$d(q-1) = y_0 + y_1 + \cdots + y_{j_d - 1} + y_{j_d}^*. \tag{2.5}$$

It is clear that $0 < y_{j_d}^* \leq y_{j_d}$ for $d \geq 1$. We define the truncation function $[y]_d$ to be the integer

$$[y]_d = y_0 + y_1 q + \cdots + y_{j_d - 1} q^{j_d - 1} + y_{j_d}^* q^{j_d}. \tag{2.6}$$

Thus, the integer $[y]_d$ is just the truncation in the q-expansion of y such that its sum of q-digits is precisely $d(q-1)$. As d varies, these numbers $[y]_d$ form a strictly increasing sequence of integers (provided these integers are defined):

$$0 = [y]_0 < [y]_1 < [y]_2 < \cdots, \quad [y]_d - [y]_{d-1} > (q-1) q^{j_{(d-1)}} > 0 \ (d \geq 1). \tag{2.7}$$

If $[y]_d$ is defined for one d, then $[y]_d$ is defined for all smaller d. In this case, we define the weight function $w_d(y)$ to be the finite integer:

$$w_d(y) = [y]_0 + [y]_1 + \cdots + [y]_d = \sum_{j=0}^{d} [y]_j. \tag{2.8}$$

If $[y]_d$ is undefined for some d, then $w_d(y) = \infty$ for this d and for all larger d.

If y is a positive integer and $s(y)$ is its sum of q-digits with $d > s(y)/(q-1)$, then the number j_d in (2.4) does not exist and $w_d(y)$ equals infinity.

Definition. For a p-adic integer $y \in \mathbf{Z}_p$, define $P(y)$ to be the convex hull in the plane $\mathbf{R}^2$ of the lattice points $(d, w_d(y))$ $(d \geq 0)$.

Lemma 2.1. The lattice points $(d, w_d(y))$ are vertices of the polygon $P(y)$ as long as $w_d(y) < \infty$. In particular, the horizontal projection of each side of $P(y)$ has length 1. The

d-th slope (i.e., the slope from $(d-1, w_{d-1}(y))$ to $(d, w_d(y))$) is given by the truncation function $[y]_d$ if $w_d(y) < \infty$.

Proof. We can assume that $w_d(y) < \infty$ and thus $[y]_d$ is defined. The slope from $(d-1, w_{d-1}(y))$ to $(d, w_d(y))$ is $w_d(y) - w_{d-1}(y) = [y]_d$ by (2.8). To prove the lemma, it suffices to prove that $[y]_d > [y]_{d-1}$ for $d \geq 1$. The last inequality follows from (2.7). The lemma is proved.

We can now state the main result whose proof is given in section 4.

Theorem 2.2. Let y be a p-adic integer. For a positive integer d, we have the following asymptotic formula for the coefficient of $(1/x)^d$ in $\zeta_y(A, x)$:

$$\sum_{h=0}^{\infty} \binom{y}{h} C_d(h) \equiv (-1)^d \prod_{\ell=1}^{d} \binom{y_{j_\ell}}{y^*_{j_\ell}} \; t^{w_d(y)} \pmod{t^{w_d(y)+1}}, \tag{2.9}$$

where the right side of (2.9) is understood to be zero if some j_ℓ (and hence j_d) is undefined. In particular, the Newton polygon of the entire function $\zeta_y(A, x)$ lies above the polygon $P(y)$.

In order to prove (2.9), we first establish an asymptotic formula for each term $C_d(h)$ in next section. The proof for each term $C_d(h)$ is reduced to a minimizing problem in integer linear programming. The proof of (2.9) is then reduced to a family of integer minimizing problems, parameterized by the integer h in (2.1). This family of minimizing problems turns out to have a unique optimal solution, namely when $h = [y]_d$. This optimal solution gives the main term in (2.9). The proof shows that the error term in (2.9) can be improved considerably. We use only the weaker error term in order to keep the arguments simpler.

Corollary 2.3. Let y be a p-adic integer. Assume that for some positive integer d, the number $j_d = j_d(y)$ is well defined and thus $j_1(y), \cdots, j_d(y)$ are all defined. Then, the following conditions are equivalent.

i). The Newton polygon of $\zeta_y(A, x)$ coincides with its lower bound $P(y)$ at the d-th vertex $(d, w_d(y))$.

ii). The first d sides of the Newton polygon of $\zeta_y(A, x)$ coincides with the first d sides of the lower bound $P(y)$.

iii). Let $y^*_{j_\ell}$ be the star function defined as in (2.5). Then,

$$\prod_{\ell=1}^{d} \binom{y_{j_\ell}}{y^*_{j_\ell}} \not\equiv \; 0 \pmod{p}. \tag{2.10}$$

Lemma 2.1 says that the horizontal projection of each side of the lower bound $P(y)$ has length 1. If the first d-sides of $\zeta_y(A, x)$ coincides with the first d-th sides of $P(y)$, then the first d zeroes of $\zeta_y(A, x)$ are simple zeroes in K and they can be arranged as $\rho_1, \cdots, \rho_d$ such that $\mathrm{ord}_t(\rho_\ell) = [y]_\ell$. In this case, the asymptotic formula (2.9) and the Weierstrass factorization of $\zeta_y(A, x)$ actually give an asymptotic formula for the roots ρ_ℓ:

$$\zeta_y(A, x) = \prod_{\ell=1}^{d} (1 - \frac{\rho_\ell}{x}) \; g_y(x), \quad \rho_\ell \equiv \binom{y_{j_\ell}}{y^*_{j_\ell}} t^{[y]_\ell} \pmod{t^{[y]_\ell + 1}}, \tag{2.11}$$

where $g_y(x)$ is an entire function in $1/x$ with no zeroes in the closed disk $\mathrm{ord}_t(x) \leq [y]_d$. The congruence for the zero ρ_ℓ in (2.11) can be seen by induction.

If the d integers y_{j_ℓ} in (2.10) are either smaller than p or equal $(q-1)$, then (2.10) holds. As an immediate consequence of Corollary 2.3, we have

Corollary 2.4. Assume that for some positive integer d, the number $j_d(y)$ is well defined and thus $j_1(y), \cdots, j_d(y)$ are all defined. Assume further that the d selected digits $\{y_{j_1}, \cdots, y_{j_d}\}$ in the q-expansion of y are either smaller than p or equal $(q-1)$. Then, the first d sides of the Newton polygon of $\zeta_y(A,x)$ coincides with the first d sides of the lower bound $P(y)$.

Corollary 2.5. If the q-digits y_j ($j \geq 1$) of y are either smaller than p or equal $(q-1)$ (this is always the case if $q = p$), then the Newton polygon of $\zeta_y(A,x)$ coincides with its lower bound $P(y)$. In this case, all zeroes of $\zeta_y(A,x)$ are in K and are simple. The zeroes of $\zeta_y(A,x)$ can be ordered as $\{\rho_1(y), \rho_2(y), \cdots,\}$ with

$$\rho_d \equiv \binom{y_{j_d}}{y^*_{j_d}} t^{[y]_d} \pmod{t^{[y]_d+1}}.$$

In particular, $\mathrm{ord}_t(\rho_d(y)) = w_d(y) - w_{d-1}(y) = [y]_d$ as long as $[y]_d$ is defined.

If y is a non-negative integer, equation (2.4) shows that $w_d(y) = \infty$ if and only if $d(q-1) > s(y)$. Thus, the polygon $P(y)$ has exactly $[s(y)/(q-1)]$ sides. Theorem 2.2 immediately gives

Corollary 2.6. If y is a positive integer and $s(y)$ is the sum of its q-digits, then $\zeta_y(A,x)$ is a polynomial in $1/x$ of degree at most $[s(y)/(q-1)]$, with equality holding if the q-digits of y are either smaller than p or equal $(q-1)$.

Example 2.7. Let n be a positive integer with $1 \leq n < q$. Let $y = (-n) = (q-n) + (q-1)q + \cdots$. One checks directly that $j_d(-1) = d-1$ and $y_{j_d}(-1) = y^*_{j_d}(-1) = (q-1)$. For $1 < n < q$, one has $j_d(-n) = d$, $y_{j_d}(-n) = (q-1)$ and $y^*_{j_d}(-n) = n-1$ for $d \geq 1$. Thus, the slopes (the truncations) are

$$[-n]_d = (q-n) + (q-1)q + \cdots + (n-1)q^d = n(q^d - 1)$$

and the weights are

$$w_d(-n) = \sum_{j=1}^{d} n(q^j - 1) = n(\frac{q}{q-1}(q^d-1) - d) = n w_d(-1).$$

Since $y_j = (q-1)$ for all $j \geq 1$, Corollary 2.5 shows that all zeroes of $\zeta_{-n}(A,x)$ are in K and are simple zero. Furthermore, these simple zeroes can be ordered as $\{\rho_1, \rho_2, \cdots\}$ with $\rho_d \equiv (-1)^{n-1} t^{n(q^d-1)} \pmod{t^{n(q^d-1)+1}}$ since $\binom{q-1}{n-1}$ is congruent to $(-1)^{n-1}$ modulo p as one can sees from the characteristic p identity $(1+x)^{q-1} = (1+x^q)(1+x)^{-1}$.

Remark. Goss first observed that Example 2.7 can be handled using the following variant of a formula of Carlitz [4, Theorem 9.4]: Let $1 \leq n < q$. Then,

$$\sum_{\text{monic } f \in A,\ 0 \leq \deg(f) \leq d} \frac{1}{f^n} = \frac{t^{w_d(-n)}}{\prod_{j=1}^{d} (t^{q^j-1} - 1)^n}. \tag{2.12}$$

It is interesting to note that conversely, Theorem 2.2 also implies Carlitz's formula. In fact, it is obvious to see that

$$\sum_{\text{monic } f\in A,\ 0\leq \deg(f)\leq d} \frac{1}{f^n} = \frac{P_n(t)}{\prod_{j=1}^{d}(t^{q^j-1}-1)^n}, \tag{2.13}$$

where $P_n(t) \in \mathbf{F}_q[t]$ is a polynomial of degree at most $w_d(-n)$ (the degree of the numerator in (2.13)). Now, Theorem 2.2 shows that $P_n(t)$ is divisible by $t^{w_d(-n)}$. Thus, there is a constant $c(n) \in \mathbf{F}_q$ such that $P_n(t) = c(n)t^{w_d(-n)}$. Setting $t = \infty$ in (2.13), one sees that $c(n) = 1$. This proves (2.12). We note that in Carlitz's original formula ([4], Theorem 9.4), the misprint $(-1)^{(k-1)m+1}$ should be $(-1)^{km}$ (they agree when $m = 1$).

Carlitz ([4], Theorem 9.4) also gives a formula for the left side of (2.12) for an arbitrary positive integer n. This formula becomes increasingly more complicated as n grows. It is no longer obvious to see how to apply it directly to determine the Newton polygon of $\zeta_y(A, x)$ for general y. If one tries to use the general Carlitz formula to prove Theorem 2.2, one would likely lead to a similar type of minimizing integer programming problems occured in our self contained proof of Theorem 2.2. Anderson [1, p505] generalized the Carlitz formula to zeta sums constructed using more general lattices of $\mathbf{F}_q((t))$. Our present case corresponds to the lattice $\frac{1}{t}\mathbf{F}_q[1/t]$.

Example 2.8. Let $y = 1/(1-q) = 1 + q + q^2 + \cdots$. Then, one checks that $j_d(y) = d(q-1)-1$ and $y_{j_d} = y^*_{j_d} = 1$. Thus, the slopes are

$$[y]_d = 1 + q + \cdots + q^{d(q-1)-1} = \frac{1}{q-1}(q^{d(q-1)} - 1)$$

and the weights are

$$w_d(y) = \sum_{j=1}^{d} \frac{1}{q-1}(q^{j(q-1)} - 1) = \frac{1}{q-1}(\frac{q^{q-1}}{q^{q-1}-1}(q^{d(q-1)} - 1) - d).$$

By Corollary 2.5, all zeroes of $\zeta_{1/(q-1)}(A, x)$ are in K and are simple. These zeroes can be ordered as $\{\rho_1, \rho_2, \cdots\}$ with $\mathrm{ord}_t(\rho_d) = \frac{1}{q-1}(q^{d(q-1)} - 1)$.

Example 2.9. Let $y = (2q-2)/(1-q^2) = (q-2) + q + (q-2)q^2 + q^3 + \cdots$ $(q > 2)$. Then, one checks that $j_d(y) = 2d-1$ and all $y^*_{j_d} = y_{j_d} = 1$ for $d \geq 1$. Thus, the slopes are

$$[y]_d = (q-2)+q+\cdots+(q-2)q^{2d-2}+q^{2d-1} = (q-2)\frac{q^{2d}-1}{q^2-1}+q\frac{q^{2d}-1}{q^2-1} = \frac{2q-2}{q^2-1}(q^{2d}-1)$$

and the weights are

$$w_d(y) = \sum_{j=1}^{d} \frac{2q-2}{q^2-1}(q^{2j} - 1) = \frac{2q-2}{q^2-1}(\frac{q^2}{q^2-1}(q^{2d} - 1) - d).$$

Since $y^*_{j_d} = 1$ for all $d \geq 1$, Corollary 2.3 shows that all zeroes of $\zeta_{(2q-2)/(1-q^2)}(A, x)$ are in K and are simple. These zeroes can be ordered as $\{\rho_1, \rho_2, \cdots\}$ with $\mathrm{ord}_t(\rho_d) = \frac{2q-2}{q^2-1}(q^{2d} - 1)$.

Remark. It is easy to see from (2.4)-(2.8) that $[y]_d \geq [-1]_d$ and $w_d(y) \geq w_d(-1)$ for each $y \in \mathbf{Z}_p$. Thus, the polygon $P(-1)$ is the minimal one among the polygons $P(y)$. That is, for every p-adic integer y, the polygon $P(y)$ lies above $P(-1)$. More general, if y_1 and y_2 are two p-adic integers such that the q-digits of y_1 are always at most the corresponding q-digits of y_2, then the polygon $P(y_1)$ lies above the polygon $P(y_2)$.

3. An Asymptotic Formula For Power Sums

The purpose of this section is to derive an asymptotic formula for the quantity $C_d(h)$ in (2.2). Given a positive integer h, we write it in q-expansion form:

$$h = h_0 + h_1 q + \cdots + h_m q^m,\ 0 \leq h_j \leq q-1,\ h_m \neq 0, \tag{3.1}$$

where m is a non-negative integer depending on h. It is easy to see directly from (2.2) that $C_d(h) = 0$ if h is not divisible by $q-1$. Let $s(h)$ be the sum of the q-digits of h. We assume that h (and therefore $s(h)$) is divisible by $q-1$. Assume further that $s(h) \geq d(q-1)$ because it will be shown that $C_d(h) = 0$ if $s(h) < d(q-1)$. Applying the construction in (2.4)-(2.5) to the positive integer h in (3.1) instead of the p-adic integer y, we get the subscript function $j_\ell = j_\ell(h)$ and the star function $h^*_{j_\ell}$ which are defined at least for $1 \leq \ell \leq d$. Define $M_d(h)$ to be the $d \times (m+1)$ matrix (which will be shown to be the optimal solution for $C_d(h)$)

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & h_{j_{d-1}} - h^*_{j_{d-1}} & h_{j_{d-1}+1} & \cdots & h_m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & h_{j_1} - h^*_{j_1} & h_{j_1+1} & \cdots & h^*_{j_2} & \cdots & 0 & 0 & \cdots & 0 \\ h_0 & h_1 & \cdots & h^*_{j_1} & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{3.2}$$

The sums of the rows of the matrix satisfy the relations:

$$\begin{aligned} (h_{j_{d-1}} - h^*_{j_{d-1}}) + h_{j_{d-1}+1} + \cdots + h_m &= (q-1)(\frac{s(h)}{q-1} - (d-1)) \geq (q-1), \\ \cdots &= \cdots \\ (h_{j_1} - h^*_{j_1}) + h_{j_1+1} + \cdots + h^*_{j_2} &= (q-1), \\ h_0 + h_1 + \cdots + h^*_{j_1} &= (q-1). \end{aligned}$$

The symbols on the d-th rows are the first few q-digits of h which sum to $(q-1)$. The symbols on the $(d-1)$-th row are the next few q-digits of h which sum to $(q-1)$, etc. Each column of $M_d(h)$ sums to the corresponding q-digit of h. For the matrix $M_d(h)$ in (3.2), we define the weight $w(M_d(h))$ of the matrix $M_d(h)$ to be the sum

$$\begin{aligned} w(M_d(h)) =& d(h_0 + h_1 q + \cdots + h^*_{j_1} q^{j_1}) + (d-1)((h_{j_1} - h^*_{j_1}) q^{j_1} + \cdots + h^*_{j_2} q^{j_2}) + \cdots \\ &+ ((h_{j_{d-1}} - h^*_{j_{d-1}}) q^{h_{j_{d-1}}} + \cdots + h_m q^m). \end{aligned} \tag{3.3}$$

That is, the (i, j)-entry of $M_d(h)$ has weight iq^j, where $1 \leq i \leq d$ and $0 \leq j \leq m$.

Example. If $h = (q-1)+(q-1)q+\cdots+(q-1)q^{d-1}$, then one sees that $s(h) = d(q-1)$ and $M_d(h)$ is just the $d \times d$ matrix

$$M_d(h) = \begin{pmatrix} 0 & 0 & \cdots & q-1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & q-1 & \cdots & 0 \\ q-1 & 0 & \cdots & 0 \end{pmatrix},$$

and the weight

$$w(M_d(h)) = d(q-1) + (d-1)(q-1)q + \cdots + q^{d-1} = \frac{q}{q-1}(q^d - 1) - d.$$

This weight is the same as the weight in Example 2.7 with $n = 1$.

Example. If $h = (q-2) + (q-1)q + (q-1)q^2 + (q-1)q^3 + \cdots + (q-1)q^{2d-2} + q^{2d-1}$, then one sees that $s(h) = (2d-1)(q-1)$ and $M_d(h)$ is just the $d \times 2d$ matrix

$$M_d(h) = \begin{pmatrix} 0 & 0 & 0 & \cdots & q-2 & q-1 & \cdots & q-1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & q-2 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ q-2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and the weight

$$w(M_d(h)) = d((q-2)+q)+(d-1)((q-2)q+q^2)+\cdots+(q-2)q^{d-1}+(q-1)q^d+\cdots+q^{2d-1}.$$

We can now state our asymptotic formula for the quantity $C_d(h)$.

Proposition 3.1. i). Let h and d be positive integers. If either h is not divisible by $(q-1)$ or $s(h)$ is smaller than $d(q-1)$, then $C_d(h) = 0$.

ii). If h is divisible by $(q-1)$ and $s(h)$ is at least $d(q-1)$, we have the asymptotic formula

$$C_d(h) \equiv (-1)^d \prod_{\ell=1}^{d-1} \binom{h_{j_\ell}}{h^*_{j_\ell}} t^{w(M_d(h))} \pmod{t^{w(M_d(h))+1}}. \tag{3.4}$$

In particular, we have the inequality

$$\mathrm{ord}_t(C_d(h)) \geq w(M_d(h)),$$

with equality holding if and only if

$$\binom{h_{j_1}}{h^*_{j_1}} \cdots \binom{h_{j_{d-1}}}{h^*_{j_{d-1}}} \not\equiv 0 \pmod{p}. \tag{3.5}$$

Remark. Part i) was first proved by Lee [8, Lemma 7-1], and reproved subsequently by other authors, see for example Gekeler [5].

Proof of Proposition 3.1: Let the h_i be the q-digits of the positive integer h as written in (3.1). One has the identity

$$(a_1t+\cdots+a_dt^d)^h=(a_1t+\cdots+a_dt^d)^{h_0}\cdots(a_1t^{q^j}+\cdots+a_dt^{dq^j})^{h_j}\cdots.$$

This identity and (2.2) show that

$$C_d(h)=\sum_{k=(k_{ij}),1\le i\le d,\ 0\le j\le m}\prod_{j=1}^{m}\binom{h_j}{k_{1j}\cdots k_{dj}}t^{w(k)}\prod_{i=1}^{d}\Big(\sum_{a_i\in\mathbf{F}_q}a_i^{\sum_j k_{ij}}\Big),$$

where the entries of the $d\times(m+1)$ matrix $k=(k_{ij})$ are non-negative integers satisfying the relation $\sum_i k_{ij}=h_j$ for each j and the weight $w(k)$ is defined to be

$$w(k)=\sum_{i=1}^{d}i\sum_{j=0}^{m}k_{ij}q^j.$$

Thus, in the definition of the weight $w(k)$, the (i,j)-entry k_{ij} has weight iq^j. Using the well known relation

$$\sum_{a\in\mathbf{F}_q}a^h=\begin{cases}0, & \text{if } (q-1)\not|h \text{ or } h=0,\\ (-1), & \text{if } (q-1)|h \text{ and } h>0,\end{cases}$$

we deduce that

$$C_d(h)=\sum_{k=(k_{ij}),1\le i\le d,\ 0\le j\le m}(-1)^d\prod_{j=1}^{m}\binom{h_j}{k_{1j}\cdots k_{dj}}t^{w(k)},\tag{3.6}$$

where the matrix $k=(k_{ij})$ satisfies the following system of equations:

$$k_{i0}+\cdots+k_{ij}+\cdots+k_{im}=(q-1)\ell_i,\ 0\le k_{ij}\le h_j,\ \ell_i\ge 1,\ 1\le i\le d,\tag{3.7}$$

and

$$k_{1j}+\cdots+k_{dj}=h_j,\ 0\le j\le m,\tag{3.8}$$

To prove part i) of Proposition 3.1, we assume that $C_d(h)\neq 0$. Then (3.7)-(3.8) has at least one matrix solution $k=(k_{ij})$. Adding the d equations in (3.7), one gets

$$s(h)=h_0+h_1+\cdots+h_m=(q-1)(\ell_1+\cdots+\ell_d)\ge(q-1)d.$$

This means that $s(h)$ is divisible by $(q-1)$ and $s(h)/(q-1)$ is at least d. The first part of Proposition 3.1 is proved.

Assume now that $s(h)$ is divisible by $(q-1)$ and $s(h)$ is at least $d(q-1)$. In this case, the matrix $M_d(h)$ in (3.2) is defined and it gives a solution of (3.7)-(3.8), which contributes to the main term in (3.4). Part ii) of Proposition 3.1 is an immediate consequence of the following lemma.

Lemma 3.2. Assume that $s(h)$ is divisible by $(q-1)$ and $s(h)$ is at least $d(q-1)$. Then for any matrix solution $k = (k_{ij})$ of (3.7)-(3.8), we have the inequality $w(k) \geq w(M_d(h))$, with equality holding if and only if the matrix solution is precisely given by the matrix $M_d(h)$ in (3.2).

Proof. This is an integer programming problem. We want to minimize the objective function $w_d(k)$ subject to the constraints in (3.7)-(3.8). To solve this problem, let $k = (k_{ij})$ be a matrix solution of (3.7)-(3.8) with $w(k)$ minimal. We need to prove that k must be the solution given by $M_d(h)$ in (3.2). The basic idea is to show that if $k = (k_{ij})$ is not $M_d(h)$, then one can apply some matrix operations to produce a new solution $k^* = (k^*_{ij})$ of (3.7)-(3.8) with $w(k^*) < w(k)$.

First, if for some $2 \leq u \leq d$, we have $\ell_u \geq 2$ in (3.7), then we can write (possibly in many ways) $k_{uj} = \alpha_{uj} + \beta_{uj}$, where $0 \leq \alpha_{uj}, \beta_{uj} \leq k_{uj}$ are integers such that $\sum_j \alpha_{uj} = (q-1)(\ell_u - 1)$ and $\sum_j \beta_{uj} = (q-1)$. Set

$$k^*_{ij} = \begin{cases} k_{ij}, & \text{if } i \neq 1, u, \\ k_{1j} + \alpha_{uj}, & \text{if } i = 1, \\ \beta_{uj}, & \text{if } i = u. \end{cases}$$

Namely, we subtract the row vector $\alpha = (\alpha_{u0}, \cdots, \alpha_{uj}, \cdots, \alpha_{um})$ from the u-th row of the matrix $k = (k_{ij})$ and add the same row vector to the first row of (k_{ij}). It is clear that the matrix $k^* = (k^*_{ij})$ is also a solution of (3.7)-(3.8) with $w(k^*) < w(k)$. In fact, by (3.9) one computes that

$$w(k) - w(k^*) = (u-1)\sum_{j=0}^{m} \alpha_{uj} q^j \geq (u-1)(q-1)(\ell_u - 1) > 0.$$

This contradicts with the minimality assumption of $w(k)$. Thus, we must have that $\ell_2 = \cdots = \ell_d = 1$.

To finish the proof of the lemma, we claim that any matrix solution k of (3.7)-(3.8) with minimal weight does not contain a submatrix of the form $\begin{pmatrix} k_{i_1 j_1} & k_{i_1 j_2} \\ k_{i_2 j_1} & k_{i_2 j_2} \end{pmatrix}$ with $k_{i_1 j_1} > 0$ and $k_{i_2 j_2} > 0$, where $i_1 < i_2$ are the row numbers and $j_1 < j_2$ are the column numbers. Otherwise, we can replace $\begin{pmatrix} k_{i_1 j_1} & k_{i_1 j_2} \\ k_{i_2 j_1} & k_{i_2 j_2} \end{pmatrix}$ by $\begin{pmatrix} k_{i_1 j_1} - 1 & k_{i_1 j_2} + 1 \\ k_{i_2 j_1} + 1 & k_{i_2 j_2} - 1 \end{pmatrix}$ and get a new matrix solution k^* with $w(k^*) < w(k)$. In fact, using (3.9) one computes that

$$w(k) - w(k^*) = (i_2 - i_1)q^{j_2} - (i_2 - i_1)q^{j_1} = (i_2 - i_1)(q^{j_2} - q^{j_1}) > 0.$$

This again contradicts with our minimality assumption of $w(k)$. The claim is proved. The claim together with the constraints $\ell_u = 1$ for $2 \leq u \leq d$ forces that any matrix solution of (3.7)-(3.8) with $w(k)$ minimal to be the solution $M_d(h)$ in (3.2). In fact, if $k = (k_{ij})$ is not the optimal matrix $M_d(h)$, one can derive a contradiction as follows. Suppose for example that $k_{d0} < h_0$. Then there is an integer $1 \leq i_1 < d$ such that $k_{i_1 0} > 0$ because $\sum_i k_{i0} = h_0$. Since $\sum_j k_{dj} = (q-1)\ell_d$, there is a positive integer j_2 such that $k_{dj_2} > 0$. This means that the matrix $k = (k_{ij})$ contains the submatrix $\begin{pmatrix} k_{i_1 0} & k_{i_1 j_2} \\ k_{d0} & k_{dj_2} \end{pmatrix}$ with $k_{i_1 0} > 0$

and $k_{dj_2} > 0$, contradicting with the claim. Thus, $k_{d0} = h_0$ and $k_{i0} = 0$ for $1 \le i < d$. Applying this argument inductively, one then sees that the d-th row of (k_{ij}) is the d-th row of $M_d(h)$. By induction, one deduces that $k = (k_{ij})$ must be the matrix $M_d(h)$. The proof of the lemma is complete.

4. Proof of Theorem

We are now in a position to prove Theorem 2.2. By (2.1)-(2.2), we need to examine the term $\binom{y}{h} C_d(h)$, where h is a positive integer. The idea is to show that the term $\binom{y}{h} C_d(h)$ contributes only to the error term in (2.9) if h is different from the truncation $[y]_d$. First note that, if one digit in the q-expansion of h is greater than the corresponding digit in the q-expansion of y, then one digit in the p-expansion of h is greater than the corresponding digit in the p-expansion of y. The well known Lucas lemma shows that $\binom{y}{h}$ is divisible by p in this case. Thus, in (2.1) we can restrict h to run only over those positive integers h whose q-digits are at most the corresponding q-digits of y. By part i) of Proposition 3.1, we can also assume that h is divisible by $(q-1)$ and $s(h)$ is at least $d(q-1)$. A positive integer h is hence called admissible with respect to y and d if h is divisible by $(q-1)$, the number $s(h)$ is at least $d(q-1)$ and the q-digits of h are at most the corresponding q-digits of y. Thus, we have shown that

$$\sum_{h=0}^{\infty} \binom{y}{h} C_d(h) = \sum_{\text{admissible } h} \binom{y}{h} C_d(h). \tag{4.1}$$

If $j_d(y)$ is undefined, then (2.4) shows that y is a non-negative integer with $s(y) < d(q-1)$. In this case, there are no admissible integers with respect to y and d. The sum in (4.1) is zero. This agrees with what Theorem 2.2 says in this case.

From now on, we assume that $j_d(y)$ is defined and thus $[y]_d$ is an admissible integer. The sum on the right side of (4.1) is now non-empty. If h is admissible, then $M_d(h)$ is well defined. The following lemma shows that if h is different from the truncation $[y]_d$, then the contribution of $\binom{y}{h} C_d(h)$ is in the error term of (2.9).

Lemma 4.1. Let d be a positive integer such that $j_d(y)$ is defined. If h is admissible with respect to y and d, then we have the inequality $w(M_d(h)) \ge w_d(y)$, with equality holding if and only if h is the admissible integer $[y]_d$ given in (2.6).

Proof. This is again some sort of integer programming problem. We need to minimize the objective function $w(M_d(h))$ with the constraint that h is admissible. Assume that h is admissible with respect to y and d such that $w(M_d(h))$ is minimal. We need to prove that $h = [y]_d$ and $w(M_d(h)) = w_d(y)$. The basic idea is to show that if h is not $[y]_d$, then we can find another admissible integer h' with the property $w(M_d(h')) < w(M_d(h))$.

Write h in the form (3.1) and construct the matrix $M_d(h)$ in (3.2) using this h. If $s(h) > d(q-1)$, then the sum of all entries in the first row of (3.2) is $s(h) - (d-1)(q-1) \ge 2(q-1)$. We can then write (possibly in many ways) the first row of $M_d(h)$ as a sum of two row vectors $\alpha = (\alpha_{10}, \cdots, \alpha_{1m})$ and $\beta = (\beta_{10}, \cdots, \beta_{1m})$ with non-negative integral coordinates such that $\sum_j \alpha_{1j} = (q-1)$ and $\sum_j \beta_{1j} = s(h) - d(q-1) \ge (q-1)$. Now, the integer

$$h' = \sum_{j=0}^{m} (h_j - \alpha_{1j}) q^j$$

is divisible by $(q-1)$ and its sum $s(h')$ of q-digits is $s(h)-(q-1) \geq d(q-1)$. Since $0 \leq h_j - \alpha_{1j} \leq h_j \leq y_j$, the integer h' is also admissible with respect to y and d. Furthermore,

$$w(M_d(h)) - w(M_d(h')) = \sum_{j=0}^{m} \alpha_{1j} q^j > 0.$$

This contradicts with our minimality assumption on weight $w(M_d(h))$. Thus, we must have $s(h) = d(q-1)$.

Next, we prove that the sum h_j of the j-th column of $M_d(h)$ in (3.2) is equal to y_j for all $0 \leq j < m$. If not, there is an integer $0 \leq j < m$ (j strictly smaller than m) such that the sum h_j of the j-th column of $M_d(h)$ in (3.2) is smaller than y_j because h is admissible and h satisfies $s(h) = d(q-1)$. Reducing the upper right entry h_m of $M_d(h)$ by one ($h_m > 0$ by (3.1)) and increasing the entry in the first row and j-th column by one, we get a new integer

$$h' = h - q^m + q^j = h_0 + \cdots + (h_j + 1)q^j + \cdots + (h_m - 1)q^m.$$

The new integer h' is divisible by $(q-1)$ and its sum $s(h')$ of q-digits is $d(q-1)$. Since $h_j + 1 \leq y_j$, the integer h' is admissible with respect to y and d. Furthermore,

$$w(M_d(h)) - w(M_d(h')) = q^m - q^j > 0.$$

This again contradicts with our minimality assumption on weight $w(M_d(h))$. Thus, the only possibility for $w(M_d(h))$ to be minimal (as h varies over the admissible integers) is that $s(h)$ equals $d(q-1)$ and the sum h_j of the j-th column of $M_d(h)$ in (3.2) equals the corresponding digit y_j of the integer in (2.6) for all $0 \leq j < m$. It follows that h is the integer $[y]_d$ defined in (2.6) and $m = j_d$. From the construction of $M_d(h)$ in (3.2) and the definition of $[y]_d$ in (2.6), one sees that $M_{d-1}([y]_{d-1})$ is just the lower left part of the larger matrix $M_d([y]_d)$. Using (3.3), one then computes that

$$w(M_d([y]_d)) - w(M_{d-1}([y]_{d-1})) = [y]_d. \tag{4.2}$$

Iterating this equation, we get

$$w(M_d([y]_d)) = [y]_0 + [y]_1 + \cdots + [y]_d = w_d(y). \tag{4.3}$$

The lemma is proved.

We now return to the proof of Theorem 2.2. By (4.1), the coefficient of $(1/x)^d$ in $\zeta_y(A, x)$ is

$$\sum_{\text{admissible } h} \binom{y}{h} C_d(h).$$

If h is different from $[y]_d$, Lemma 4.1 shows that the contribution of the term $\binom{y}{h} C_d(h)$ is in the error term of (2.9). Thus, it remains to show

$$\binom{y}{[y]_d} C_d([y]_d) \equiv (-1)^d \prod_{\ell=1}^{d} \binom{y_{j_\ell}}{y_{j_\ell}^*} t^{w_d(y)} \pmod{t^{w_d(y)+1}}. \tag{4.4}$$

Since the q-digits of $[y]_d$ (except for the last digit $y^*_{j_d}$) is the same as the corresponding q-digits of y, the Lucas lemma shows that

$$\binom{y}{[y]_d} \equiv \binom{y_{j_d}}{y^*_{j_d}} \pmod{p}. \tag{4.5}$$

Lemma 4.1 shows that $w(M_d([y]_d)) = w_d(y)$. Applying Proposition 3.1, we deduce that

$$C_d([y]_d) \equiv (-1)^d \prod_{\ell=1}^{d-1} \binom{y_{j_\ell}}{y^*_{j_\ell}} t^{w_d(y)} \pmod{t^{w_d(y)+1}}. \tag{4.6}$$

This together with (4.5) gives (4.4). The proof is complete.

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