Poles of Zeta Functions of Complete Intersections

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Abstract. A vanishing theorem is proved for ℓ-adic cohomology with compact support on a singular affine complete intersection. As an application, it is shown that for an affine complete intersection defined over a finite field of q elements, the reciprocal "poles" of the zeta function are always divisible by q as algebraic integers. A p-adic proof is also given, which leads to further q-divisibility of the poles or equivalently an improvement of the polar part of the Ax-Katz theorem for an affine complete intersection. Similar results hold for a projective complete intersection.

1. Introduction

Let $\mathbf{F}_q$ be a finite field of q elements with characteristic p. Let X be an n-dimensional algebraic set defined over $\mathbf{F}_q$. The zeta function of $X/\mathbf{F}_q$ is defined by

$$\begin{aligned} Z(X,T) &= \prod_{x\in X_0} \frac{1}{1-T^{\deg(x)}} \\ &= \exp(\sum_{d=1}^{\infty} \frac{T^d}{d} \#X(\mathbf{F}_{q^d})), \end{aligned}$$

where X_0 denotes the set of closed points of $X/\mathbf{F}_q$ and $\#X(\mathbf{F}_{q^d})$ denotes the number of $\mathbf{F}_{q^d}$-rational points on X. It is easy to see that $Z(X,T)$ is a power series with integer coefficients. Dwork's rationality theorem [Dw] shows that the zeta function $Z(X,T)$ is rational in T. Thus, there are algebraic integers $\rho_1, \cdots, \rho_r, \beta_1, \cdots, \beta_s$ such that

$$Z(X,T)^{(-1)^{n-1}} = \frac{\prod_{i=1}^{r}(1-\rho_i T)}{\prod_{j=1}^{s}(1-\beta_j T)}.$$

There is a good reason that we modified the above zeta function by the power $(-1)^{n-1}$. The ρ_i are called the reciprocal zeroes. The β_j are called the reciprocal poles. Deligne's theorem on Riemann hypothesis [De2] shows that the absolute values of the ρ_i and the β_j are integral powers of $\sqrt{q}$.

In this article, we are interested in entireness properties of $Z(X,T)^{(-1)^{n-1}}$. Namely, we hope to say some "vanishing" information about the reciprocal poles β_j. Ideally, one would like $Z(X,T)^{(-1)^{n-1}}$ to be essentially a polynomial except for possible trivial poles. This is indeed the case if X is sufficiently nice. The following theorem is well known and follows from the weak Lefschetz theorem and the Poincare duality in ℓ-adic cohomology developed in mid-sixties.

Theorem 1.1a (projective strong entireness). *If X is an n-dimensional smooth projective complete intersection defined over $\mathbf{F}_q$, then*

$$Z(X,T)^{(-1)^{n-1}} = \frac{P(T)}{\prod_{i=0}^{n}(1-q^iT)^{(-1)^{n-1}}},$$

where $P(T)$ is a polynomial with integer coefficients.

The hypersurface case of this theorem was first proved by Dwork in early sixties using p-adic methods. The general case can also be proved using Dwork's methods, as shown recently by Adolphson-Sperber [AS2]. This theorem and the excision sequence of ℓ-adic cohomology immediately imply the following affine version which can be traced back to mid-sixties, see also [AS2] for a recent p-adic proof.

Theorem 1.1b (affine strong entireness). *If X is an n-dimensional affine complete intersection such that both its projective closure and its infinity part are a projective smooth complete intersection over $\mathbf{F}_q$, then*

$$Z(X,T)^{(-1)^{n-1}} = \frac{P(T)}{(1-q^nT)^{(-1)^{n-1}}},$$

where $P(T)$ is a polynomial with integer coefficients.

Unfortunately, the above strong entireness does not hold in singular cases. Motivated by Bombieri's observation [Bo, Theorem 1.1] and Dwork's question [DGS, Remark 7.5 in Chapter II], it would be interesting to say some non-trivial information about the reciprocal poles β_j even if X is a singular complete intersection. In this direction, we shall prove the following result here.

Theorem 1.2a (projective weak entireness). *If X is an n-dimensional projective set theoretic complete intersection over $\mathbf{F}_q$, then*

$$Z(X,T)^{(-1)^{n-1}} = \frac{P(T)}{(1-T)^{(-1)^{n-1}}\prod_{j=1}^{s}(1-q\gamma_j T)},$$

where $P(T)\in\mathbf{Z}[T]$ and the γ_j are algebraic integers. Namely, the non-trivial reciprocal poles β_j $(\neq 1)$ are divisible by q as algebraic integers.

If X^{aff} denotes the affine cone of a projective variety X, then it is easy to see that

$$Z(X^{\text{aff}},T) = \frac{Z(X,qT)}{(1-T)Z(X,T)}.$$

Thus, Theorem 1.2a is an immediate consequence of the following affine version.

Theorem 1.2b (affine weak entireness). *If X is an n-dimensional affine set theoretic complete intersection over $\mathbf{F}_q$, then*

$$Z(X,T)^{(-1)^{n-1}} = \frac{P(T)}{\prod_{j=1}^{s}(1-q\gamma_j T)},$$

where $P(T)\in\mathbf{Z}[T]$ and the γ_j are algebraic integers. Namely, the reciprocal poles β_j are divisible by q as algebraic integers.

Note that the weak entireness is false if we drop the complete intersection condition. For example, consider the affine variety

$$X = \text{Spec } \mathbf{F}_q[X_1,X_2,X_3,X_4]/(X_1X_3, X_1X_4, X_2X_3, X_2X_4). \tag{1.1}$$

It is easy to see that X is just the union in $\mathbf{A}^4$ of the two planes $X_1 = X_2 = 0$ and $X_3 = X_4 = 0$ intersecting at the origin. One computes that $Z(X,T) = (1-T)/(1-q^2T)^2$

and $Z(X,T)^{(-1)^{2-1}} = (1-q^2T)^2/(1-T)$. Thus, the above weak entireness fails for this connected affine surface X.

Remark. For L-functions of exponential sums on the affine (or toric) n-space, the analogue of Theorem 1.2b is an immediate consequence of Dwork's trace formula. Although the zeta function $Z(X,T)$ can be expressed in terms of L-functions of exponential sums on an affine space, Theorem 1.2b cannot be derived directly from the exponential sum version. Theorem 1.2b is more subtle and involves cancellation or divisibility of functions, see section 3.

We shall give two proofs of Theorem 1.2b. Both are reduced to proving a vanishing theorem on cohomology. The first approach is conceptually very simple, uses Deligne's integrality theorem and some standard properties of ℓ-adic cohomology. It works in greater generality, see section 2 for an extension of the above weak entireness theorem to certain L-functions over more general varieties. The second approach, which is how we first proved Theorem 1.2b, uses Dwork's p-adic method. As Ogus pointed out, one could also use the rigid cohomology developed by Berthelot, which is a significant generalization of the Dwork-Monsky theory. The argument, entirely cohomological, would be similar to the proof for ℓ-adic case. We shall, however, contend with the much simpler (although somewhat less general) setting as studied by Dwork. This simpler p-adic approach is more explicit. It has the advantage to be able to improve Theorem 1.2b further and also to improve the polar part of the Ax-Katz theorem in the complete intersection case. The author does not know how to obtain this improvement by using ℓ-adic approach.

To recall the Ax-Katz theorem [Ax][Ka1], let X be an affine algebraic set defined by the vanishing of r polynomials in N variables of degrees $d_1, \cdots, d_r$. Let μ be the smallest *non-negative* integer which is greater than or equal to

$$\frac{N - \sum_{i=1}^{r} d_i}{\max_i d_i}.$$

The Ax-Katz theorem says that all reciprocal zeroes ρ_i and all reciprocal poles β_j are divisible by q^μ as algebraic integers. Since this theorem is best possible in general for each N and each multi-degree $\{d_1, \cdots, d_r\}$, it would be unreasonable to expect uniform improvements of q-divisibility for all the α_i and β_j in general. However, in various special cases, improvements have been made by Adolphson-Sperber [AS1] if one takes into account the terms actually appearing in the f_i and by Moreno-Moreno [MM] if one takes into account the p-digits of the exponents of the terms actually appearing in the f_i. In the latter case, the characteristic p is necessarily small, say $p < \max d_i$, in order for an improvement to occur.

Now, if X is an affine complete intersection, the above weak entireness theorem shows that all the reciprocal poles β_j are already divisible by q as algebraic integers. This suggests that it might be reasonable to expect an improvement of the polar part of the Ax-Katz theorem for complete intersections. We shall show that this is indeed the case in some sense. Recall that if X is a complete intersection of co-dimension r in $\mathbf{A}^N$, Macaulay's unmixedness theorem [Ku, p187] shows that one can choose a *regular sequence* of polynomials $f_1, \cdots, f_r$ of $\mathbf{F}_q[x_1, \cdots, x_N]$ such that X is defined by the vanishing of the set $\{f_1, \cdots, f_r\}$. The sequence $\{f_1, \cdots, f_r\}$ being a regular sequence means that, for each $1 \leq i \leq r$, the multiplication map f_i on the ring $\mathbf{F}_q[x_1, \cdots, x_N]/(f_1, \cdots, f_{i-1})$ is injective.

Theorem 1.3. *Let X be a complete intersection in $\mathbf{A}^N$ defined by the vanishing of a regular sequence of polynomials $\{f_1, \cdots, f_r\}$ over $\mathbf{F}_q$ in N variables of degrees $d_1, \cdots, d_r$. Let ν be the smallest non-negative integer which is greater than or equal to*

$$\frac{(N - 1 - \sum_{i=1}^r d_i)}{\max_i d_i}.$$

Then, all the reciprocal poles β_j of $Z(X,T)^{(-1)^{N-r-1}}$ are divisible by $q^{1+\nu}$ as algebraic integers.

Since $1 + \nu \geq 1$, Theorem 1.3 is an improvement of Theorem 1.2b. It is easy to see that

$$1 + \nu = \begin{cases} \mu, & \text{if } (N - 1 - \sum_i d_i)/\max d_i \text{ is a non-negative integer,} \\ 1 + \mu, & \text{otherwise.} \end{cases}$$

Thus, Theorem 1.3 is also an improvement of the polar part of the Ax-Katz theorem in the case of affine complete intersections. We shall show that Theorem 1.3 is best possible for some hypersurfaces of degree d in $\mathbf{A}^N$ provided that d is greater than 2 and d does not divide $(N-1)(N-2)$. Presumably, Theorem 1.3 has a direct Hodge-theoretic analogue, improving earlier results of Deligne-Dimca etc [DD] in this direction. Of course, Theorem 1.3 gives new information only for singular complete intersections. For a sufficiently smooth complete intersection, much more precise information about the poles is already given in Theorem 1.1b.

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2. Poles of L-functions: ℓ-adic methods

In this section, we use ℓ-adic cohomology and Deligne's integrality theorem to prove a more general weak entireness result. The main idea is to use the excision sequence to prove a vanishing theorem on affine complete intersections. The vanishing theorem also implies a weak Lefschetz theorem for singular projective complete intersections. A quick summary of the basic properties for ℓ-adic cohomology is given in Katz [Ka2].

Let X be a separated scheme of finite type over a field K. Let ℓ be a prime number different from the characteristic of K. Let $\mathcal{E}$ be a constructible ℓ-adic sheaf on X. We shall use $H_c^i(X, \mathcal{E})$ to denote the i-th ℓ-adic cohomology group with compact support. The restriction $\mathcal{E}|U$ of a sheaf $\mathcal{E}$ to a subscheme U will also be denoted by $\mathcal{E}$.

Lemma 2.1. *Let X be an n-dimensional set theoretic complete intersection in some affine, smooth and equi-dimensional Y of finite type over K. Suppose that $\mathcal{E}$ is an lisse ℓ-adic sheaf on Y. Then for all $i < n$, we have*

$$H_c^i(X, \mathcal{E}) = 0.$$

Proof. We may assume that X is an ideal theoretic complete intersection defined by the vanishing of r regular functions on Y with $\dim(X) = \dim(Y) - r$. We prove the lemma using induction on r.

If $r = 0$, then $X = Y$ is smooth, affine and equi-dimensional. The Poincare duality (valid for smooth, geometrically connected X and lisse $\mathcal{E}$) shows that as ℓ-adic vector spaces,

$$\dim H_c^i(X, \mathcal{E}) = \dim H^{2n-i}(X, \mathcal{E}),$$

where H^j denotes j-th ℓ-adic cohomology without support. On the other hand, the Lefschetz affine theorem (valid for affine X and any constructible $\mathcal{E}$) shows that

$$H^j(X, \mathcal{E}) = 0 \ (j > n),$$

Thus, the lemma is true for $r = 0$ (this case is well known).

Suppose now that the lemma is true for all complete intersections which can be defined by less than $r(> 0)$ equations in a smooth affine equi-dimensional scheme of finite type. Assume now that X is a complete intersection defined by the vanishing of r equations $\{f_1, \cdots, f_r\}$ on some smooth affine equi-dimensional Y of finite type. Since Y is Cohen-Macaulay, by re-choosing the generators if necessary, we may assume that the sequence $\{f_1, \cdots, f_r\}$ forms a regular sequence of the ring of regular functions on Y. In particular, for each $1 \leq i \leq r$, the subsequence $\{f_1, \cdots, f_i\}$ also defines a complete intersection in Y.

Let Z be the affine complete intersection in Y defined by the vanishing of the first $r-1$ functions $\{f_1, \cdots, f_{r-1}\}$. Let U be the compliment of X in Z. The variety U is just the localization of Z at f_r and thus U is the complete intersection defined by the vanishing of the $r-1$ functions $\{f_1, \cdots, f_{r-1}\}$ in the smooth affine equi-dimensional Y_r, where Y_r is the localization of Y at f_r. The sheaf $\mathcal{E}$ is clearly lisse on both Z and U as it is lisse on the ambient space Y. By induction, we have

$$H_c^i(Z, \mathcal{E}) = 0 \ (i < \dim(Z)), \quad H_c^j(U, \mathcal{E}) = 0 \ (j < \dim(Z)). \tag{2.1}$$

Since U is open in Z with closed complement X, we have the excision long exact sequence for ℓ-adic cohomology with compact support:

$$H_c^i(Z,\mathcal{E}) \longrightarrow H_c^i(X,\mathcal{E}) \longrightarrow H_c^{i+1}(U,\mathcal{E}) \longrightarrow H_c^{i+1}(Z,\mathcal{E}).$$

This exact sequence and (2.1) together show that

$$H_c^i(X,\mathcal{E}) = 0$$

for all $i < \dim(Z) - 1 = \dim(X)$. The proof is complete.

Lemma 2.2. *Let X be an n-dimensional projective set theoretic complete intersection in some projective, smooth and equi-dimensional Y over K. Let $\mathcal{E}$ be an lisse ℓ-adic sheaf on Y. Then the natural map*

$$H^i(Y,\mathcal{E}) \longrightarrow H^i(X,\mathcal{E})$$

is an isomorphism for all $i < n$ and injective for $i = n$.

Proof. This is a consequence of Lemma 2.1. We may assume that X is defined by the vanishing of a regular sequence $\{f_1, \cdots, f_r\}$ of elements in the homogeneous coordinate ring of Y with the same degree. For $0 \le j \le r$, let Y_j be the projective complete intersection in Y defined by the vanishing of $\{f_1, \cdots, f_j\}$. Thus, we have

$$X = Y_r \subset Y_{r-1} \subset \cdots \subset Y_0 = Y.$$

Let U_j be the compliment $Y_j - Y_{j+1}$ of Y_{j+1} in Y_j. It is easy to see that U_j is the complete intersection in the smooth affine and equi-dimensional $Y - \{f_{j+1} = 0\}$ defined by the vanishing of the regular functions $\{f_1/f_{j+1}, \cdots, f_j/f_{j+1}\}$ on $Y - \{f_{j+1} = 0\}$. Lemma 2.1 shows that

$$H_c^i(U_j,\mathcal{E}) = 0, \text{ for } i < \dim(Y_j).$$

This and the long exact excision sequence

$$H_c^i(U_j,\mathcal{E}) \longrightarrow H^i(Y_j,\mathcal{E}) \longrightarrow H^i(Y_{j+1},\mathcal{E}) \longrightarrow H_c^{i+1}(U_j,\mathcal{E})$$

show that the natural map

$$H^i(Y_j, \mathcal{E}) \longrightarrow H^i(Y_{j+1}, \mathcal{E})$$

is an isomorphism for $i < \dim(Y_{j+1})$ and injective for $i = \dim(Y_{j+1})$. The proof is complete.

Remark. The condition in Lemma 2.2 can be weakened somewhat. For example, the above proof shows that the condition that $\mathcal{E}$ is lisse on Y can be weaken to the condition that $\mathcal{E}$ is lisse on the compliment $Y - S$, where S is the hypersurface in Y defined by the vanishing of the product $f_1 \cdots f_r$. Lemma 2.2 can be viewed as a weak Lefschetz theorem for complete intersections. It is well known if X is a non-singular complete intersection in Y. The general case does not seem to have been noted before.

We now return to L-functions over finite fields and prove a result more general than Theorem 1.2b. Let $\mathbf{F}_q$ be the finite field of q elements with characteristic p. Let X be an affine scheme of finite type over $\mathbf{F}_q$. Let $\mathcal{E}$ be a constructible ℓ-adic sheaf on X. Following Deligne [De1], an ℓ-adic sheaf $\mathcal{E}$ is called integral if for each closed point $x \in X_0$, the eigenvalues of the geometric Frobenius Frob_x acting on the vector space $\mathcal{E}_x$ are algebraic integers in a fixed number field. Assume that $\mathcal{E}$ is integral. The L-function of the sheaf $\mathcal{E}$ on X is defined by

$$L(\mathcal{E}/X, T) = \prod_{x \in X_0} \frac{1}{\det(I - \mathrm{Frob}_x T^{\deg(x)})} \in 1 + T\mathcal{O}[[T]],$$

where $\mathcal{O}$ denotes the ring of integers in the given number field. Grothendieck's rationality theorem [Gr] shows that $L(\mathcal{E}/X, T)$ is a rational function in T.

Theorem 2.3 (weak entireness). *Let X be a set theoretic complete intersection in some affine, smooth and equi-dimensional Y of finite type over $\mathbf{F}_q$. Suppose that $\mathcal{E}$ is an integral and lisse ℓ-adic sheaf on Y. Then*

$$L(\mathcal{E}/X, T)^{(-1)^{\dim(X)-1}} = \frac{P(T)}{\prod_{j=1}^{s}(1 - q\gamma_j T)},$$

where $P(T)$ is a polynomial in $\mathcal{O}[T]$ and the γ_j are algebraic integers.

Proof. Let $n = \dim(X)$. Let $\bar{\mathbf{F}}_q$ be a fixed algebraic closure of $\mathbf{F}_q$. The ℓ-adic cohomological formula [Gr] gives

$$L(\mathcal{E}/X, T) = \prod_{i=0}^{2n} \det(I - FT | H_c^i(X \otimes_{\mathbf{F}_q} \bar{\mathbf{F}}_q, \mathcal{E}))^{(-1)^{i-1}},$$

where F is the geometric Frobenius F acting on $H_c^i(X \otimes_{\mathbf{F}_q} \bar{\mathbf{F}}_q, \mathcal{E})$. Deligne's integrality theorem [De1, Theorem 5.2.2] shows that for $i > n$, q^{i-n} divides the eigenvalues of F on $H_c^i(X \otimes_{\mathbf{F}_q} \bar{\mathbf{F}}_q, \mathcal{E})$ as algebraic integers. On the other hand, Lemma 2.1 shows that $H_c^i(X \otimes_{\mathbf{F}_q} \bar{\mathbf{F}}_q, \mathcal{E}) = 0$ for $i < n$. The proof is complete.

Taking $\mathcal{E}$ to be the constant sheaf $\mathbf{Q}_\ell$ on Y, we obtain the following corollary which generalizes Theorem 1.2b.

Corollary 2.4. *Let X be a set theoretic complete intersection in some affine, smooth and equi-dimensional Y of finite type over $\mathbf{F}_q$. Then*

$$Z(X, T)^{(-1)^{\dim(X)-1}} = \frac{P(T)}{\prod_{j=1}^{s}(1 - q\gamma_j T)},$$

where $P(T)$ is a polynomial in $\mathbf{Z}[T]$ and the γ_j are algebraic integers.

For another example, we consider the L-function attached to the exponential sum of the type

$$\sum_{f(x)=0,\ g(x)\neq 0} \Psi(\mathrm{tr}(\frac{h(x)}{g(x)})),$$

where Ψ is a non-trivial additive character of $\mathbf{F}_q$ and the f, g, h are pairwise prime polynomials in N variables over $\mathbf{F}_q$. The space X is just the hypersurface $\{f = 0\}$ contained in the smooth connected affine $Y = \mathbf{A}^N - \{g \neq 0\}$. The associated ℓ-adic sheaf on X is clearly the restriction to X of an lisse ℓ-adic sheaf on Y. Thus, Theorem 2.3 applies to this example.

Remark. If X is the n-dimensional torus $\mathbf{G}_m^n$, one can say even more about the reciprocal poles $q\gamma_j$ if $\mathcal{E}$ gives rise a continuous complex (or more general overconvergent

p-adic) representation of the arithmetic fundamental group $\pi_1^{\mathrm{arith}}(X/\mathbf{F}_q)$. In fact, Dwork trace formula shows that each γ_j is of the form $q^i\rho_j$ for some integer $i \geq 0$, where ρ_j is a reciprocal zero of $P(T)$, the numerator of $L(\mathcal{E}/X,T)^{(-1)^{\dim(X)-1}}$. See Bombieri [Bo, Theorem 1.1] for more details in the case of L-functions of exponential sums over the torus $\mathbf{G}_m^n$.

It would be interesting to know if the assumption in Theorem 2.3 that $\mathcal{E}$ is lisse on Y can be weakened to the condition that $\mathcal{E}$ is lisse on X. A positive answer would imply that the L-function of any complex continuous representation on any equi-dimensional affine curve over $\mathbf{F}_q$ (automatically a set theoretic complete intersection by Cowsik-Nori [CN]) is always a polynomial up to some trivial reciprocal poles of the form $q\gamma_j$, where the γ_j are roots of unity. This consequence is of course well known if X is smooth, but seems open if X is singular. As a partial evidence, a characteristic p version of our question was raised in [TW1] and answered positively in [TW2]. See the survey in [Wa] for related questions.

3. Poles of zeta functions: p-adic methods

In this section, we study the poles of the zeta function of an affine complete intersection using p-adic methods. This is the way we first proved Theorem 1.2b.

It is well known that the zeta function can be expressed as an alternating product of the Fredholm determinant of certain completely continuous Frobenius operator acting on a certain complex of p-adic Banach spaces. By induction, we show that the eigenvalues of the Frobenius operator acting on each piece of the complex are algebraic integers. To obtain strong entireness result, one needs to prove that the complex is acyclic in positive dimensions. Unfortunately, the complex is not acyclic in general, unless X is a sufficiently smooth complete intersection [AS2]. However, to obtain weaker result stated in Theorem 1.2b, we shall show that it is sufficient to restrict to a subcomplex and this subcomplex is acyclic in positive dimensions for X satisfying the assumption of Theorem 1.2b. By keeping track of the p-adic norm of the Frobenius operator acting on various pieces of the total complex, one then obtains Theorem 1.3

Let X be an affine algebraic set in $\mathbf{A}^N$, defined by the vanishing of r polynomials $f_i(x_1,\cdots,x_N)$ $(1\leq i\leq N)$ over $\mathbf{F}_q$. We do not assume that X is a complete intersection at this point. To use Dwork's trace formula, we need to express the zeta function of X in terms of the L-function of the exponential sums associated to the polynomial

$$g = x_{N+1}f_1(x_1,\cdots,x_N)+\cdots+x_{N+r}f_{N+r}(x_1,\cdots,x_N).$$

Fix a non-trivial additive character Ψ of $\mathbf{F}_q$ with complex values, let $\Psi_k=\Psi\circ\mathrm{tr}_k$ be the induced additive character of $\mathbf{F}_{q^k}$, where tr_k denotes the trace from $\mathbf{F}_{q^k}$ to $\mathbf{F}_q$ and k denotes a positive integer. For an $\mathbf{F}_q$-regular function h on an $\mathbf{F}_q$-variety V, define

$$S_k(V,h)=\sum_{x\in V(\mathbf{F}_{q^k})}\Psi_k(h(x)),$$

$$L(V,h,T)=\exp(\sum_{k=1}^{\infty}S_k(V,h)\frac{T^k}{k}).$$

It is easy to see that

$$S_k(\mathbf{A}^{N+r},g,T) \;=\; q^r\#X(\mathbf{F}_{q^k}).$$

Thus,

$$Z(X,q^rT)=L(\mathbf{A}^{N+r},g,T). \tag{3.1}$$

Let Δ_1 be the convex closure in $\mathbf{R}^{N+r}$ of the origin and the exponents of non-zero monomials in g. Normally, it would be natural to work with Δ_1. However, we find it easier later to work with a larger polyhedron Δ. For this purpose, we let Δ be the convex closure in $\mathbf{R}^{N+r}$ of Δ_1 and the unit coordinate vectors e_i $(i=1,\cdots,N+r)$, where e_i is the unit coordinate vector on the x_i-axis. Let $C(\Delta)$ be the real cone generated by Δ, i.e., the collection of all non-negative real multiples of points of Δ. It is clear that $C(\Delta)$ is just the cone in $\mathbf{R}^{N+r}$ with all coordinates non-negative (the first quadrant). The cone $C(\Delta_1)$ generated by Δ_1 is a subcone of $C(\Delta)$. For a point $u=(u_1,\cdots,u_{N+r})\in C(\Delta)$, define its weight $w(u)$ to be the smallest real number δ such that the dilation $\delta\Delta$ contains u. In the special case that $u = (u_1, \cdots, u_{N+r})$ is in the smaller cone $C(\Delta_1)$, we have $w(u) = u_{N+1} + \cdots + u_{N+r}$.

Let $\Omega_0 = \mathbf{Q}_p(\zeta_p, \zeta_{q-1})$, where $\mathbf{Q}_p$ denotes the p-adic rational numbers and ζ_p (resp. ζ_{q-1}) denotes a primitive p-th (resp. $(q-1)$-th) root of unity. Let $\mathcal{O}_0$ be the ring of integers in Ω_0 and let $\pi \in \mathcal{O}_0$ be a uniformizing parameter, so $\mathrm{ord}_p \pi = 1/(p-1)$. Define two p-adic Banach spaces

$$B = \{ \sum_{u \in \mathbf{Z}^{N+r} \cap C(\Delta)} A_u \pi^{w(u)} x^u | A_u \in \Omega_0, A_u \to 0 \text{ as } u \to \infty \},$$

$$B_1 = \{ \sum_{u \in \mathbf{Z}^{N+r} \cap C(\Delta_1)} A_u \pi^{w(u)} x^u | A_u \in \Omega_0, A_u \to 0 \text{ as } u \to \infty \}.$$

The reason that we choose to work with B instead of B_1 in this section is that the "reduction" modulo π of B is the full polynomial ring $\mathbf{F}_q[x_1, \cdots, x_{N+r}]$ instead of a smaller conical ring, and thus we will be in a more familiar situation.

Dwork's trace formula for the torus $\mathbf{G}_m^{N+r}$ shows that there is a completely continuous Ω_0-linear operator α of B such that

$$L(\mathbf{G}_m^{N+r}, g, T)^{(-1)^{N+r-1}} = \prod_{l=0}^{N+r} \det(I - Tq^l \alpha | B)^{(-1)^l \binom{N+r}{l}}. \tag{3.2}$$

The subspace B_1 is α-stable. Furthermore, the operator α is easily seen to be topologically nilpotent on the quotient space B/B_1. Thus, the Fredholm determinants of α acting on both B and B_1 are the same. In particular, one could use B_1 in place of B in (3.2). That is, we have

$$L(\mathbf{G}_m^{N+r}, g, T)^{(-1)^{N+r-1}} = \prod_{l=0}^{N+r} \det(I - Tq^l \alpha | B_1)^{(-1)^l \binom{N+r}{l}}. \tag{3.3}$$

We shall work with the larger space B in this section and restrict to B_1 in next section.

Formula (3.2) has a homological version. Let $S = \{1, \cdots, N+r\}$. There are commuting differential operators D_i ($i \in S$) of B such that

$$\alpha \circ D_i = q D_i \circ \alpha. \tag{3.4}$$

Let $K_{\cdot}$ be the Koszul complex

$$0 \to K_{N+r} \to K_{N+r-1} \to \cdots \to K_1 \to K_0 \to 0 \tag{3.5}$$

on B defined by $D_1, \cdots, D_{N+r}$. Its component of degree l is given by

$$K_l = \bigoplus_{|J|=l} Be_J,$$

where the sum is over all subsets J of S of cardinality l and e_J is a formal symbol. Define an endomorphism $\alpha_l : K_l \to K_l$ by

$$\alpha_l = \bigoplus_{|J|=l} q^l \alpha.$$

By (3.4), these α_l induce a chain map $\alpha_{\cdot}$ on the complex in (3.5). By (3.2), one gets the chain level form of the Dwork trace formula for the torus $\mathbf{G}_m^{N+r}$:

$$L(\mathbf{G}_m^{N+r}, g, T)^{(-1)^{N+r-1}} = \prod_{l=0}^{N+r} \det(I - T\alpha_l | K_l)^{(-1)^l}. \tag{3.6}$$

What we really need is the slightly more subtle trace formula for the affine space $\mathbf{A}^{N+r}$. To describe it, recall that $S = \{1, \cdots, N+r\}$. For a subset J of S, let B^J denote the space of those power series in B which are divisible by all x_j with $j \in J$. Each B^J is stable under the operator α. Let $K_{\cdot}(S)$ be the subcomplex of $K_{\cdot}$ defined by

$$K_l(S) = \bigoplus_{J \subseteq S, |J|=l} B^{S-J} e_J. \tag{3.7}$$

Then, the trace formula for the affine $(N+r)$-space is given by

$$L(\mathbf{A}^{N+r}, g, T)^{(-1)^{N+r-1}} = \prod_{l=0}^{N+r} \det(I - T\alpha_l | K_l(S))^{(-1)^l}. \tag{3.8}$$

Combining (3.1), (3.8) and (3.7), we deduce with the change of symbol $S - J \to J$ that

$$Z(X, q^r T)^{(-1)^{N+r-1}} = \prod_{J \in S} \det(I - Tq^{N+r-|J|}\alpha | B^J)^{(-1)^{N+r-|J|}}. \tag{3.9}$$

Let $S_1 = \{1, \cdots, N\}$ and $S_2 = \{N+1, \cdots, N+r\}$. The variables in S_1 and S_2 play somewhat different role. We need to treat them separately. For $J \subseteq S$, we can uniquely write $J = \{J_1, J_2\}$ with $J_1 \subseteq S_1$ and $J_2 \subseteq S_2$. In this notation, we also write B^J as B^{J_1,J_2}. Equation (3.9) can be rewritten as

$$Z(X, q^r T)^{(-1)^{N+r-1}} = \prod_{J_1,J_2} \det(I - Tq^{N+r-|J_1|-|J_2|}\alpha|B^{J_1,J_2})^{(-1)^{N+r-|J_1|-|J_2|}}. \qquad (3.10)$$

To prove Theorem 1.2b, the first step is to eliminate those factors in (3.10) with $|J_1| < N$. For this purpose, we need the following result.

Lemma 3.1. *The eigenvalues of α acting on B^{J_1,J_2} are algebraic integers, and are divisible by $q^{|J_2|}$ as algebraic integers.*

Proof. Since B^{J_1,J_2} is an α-stable subspace of B^{J_2}, it suffices to prove the lemma for B^{J_2}. We use induction on $|J_2|$.

If $|J_2| = 0$, Möbius inversion of equation (3.2) shows that the eigenvalues of α acting on B are algebraic integers since the reciprocal zeroes and poles of L-functions are algebraic integers. This proves the lemma for the case $|J_2| = 0$.

Assume now that $|J_2| > 0$. Let X^{J_2} be the affine variety in $(N+r-|J_2|)$-dimensional space $\mathbf{G}_m^N \times \mathbf{G}_m^{r-|J_2|}$ defined by the vanishing of the $|J_2|$ polynomials $f_j(x_1, \cdots, x_N)$ $(N+j \in J_2)$, where $\mathbf{G}_m^{r-|J_2|}$ corresponds to the free toric variables x_i with $i \in S_2 - J_2$. Let

$$g_{J_2} = \sum_{N+j \in J_2} x_{N+j} f_j(x_1, \cdots, x_N).$$

Similar to (3.1), one finds that

$$Z(X^{J_2}, q^{|J_2|}T) = L(\mathbf{G}_m^{N+r-|J_2|} \times \mathbf{A}^{|J_2|}, g_{J_2}, T), \qquad (3.11)$$

where $\mathbf{A}^{|J_2|}$ corresponds to the variables x_{N+j} $(N+j \in J_2)$ in g_{J_2}. Let J_2^c be the total compliment $S - J_2$ of J_2 in S. Applying a slightly more general trace formula to the right side of (3.11), one obtains the formula

$$Z(X^{J_2}, q^{|J_2|}T)^{(-1)^{N+r-1}} = \prod_{I_1 \subseteq J_2^c, I_2 \subseteq J_2} \det(I - Tq^{N+r-|I_1|-|I_2|}\alpha|B^{I_2})^{(-1)^{N+r-|I_1|-|I_2|}}. \qquad (3.12)$$

Note that the space B^{I_2} in the above formula is independent of I_1. By induction, for $|I_2| < |J_2|$, the eigenvalues of $q^{N+r-|I_1|-|I_2|}\alpha$ acting on B^{I_2} are algebraic integers, and divisible as algebraic integers by $q^{N+r-|I_1|}$. Since $N+r-|I_1| \geq |J_2|$, it follows from (3.12) that for the p-adic meromorphic function (the part in (3.12) with $I_2 = J_2$) defined by

$$F(J_2, T) = \prod_{I_1 \subseteq J_2^c} \det(I - Tq^{N+r-|I_1|-|J_2|}\alpha|B^{J_2})^{(-1)^{N+r-|I_1|-|J_2|}}, \tag{3.13}$$

its reciprocal zeroes and poles are algebraic integers and divisible as algebraic integers by $q^{|J_2|}$. Möbius inversion of (3.13) shows that the eigenvalues of α acting on B^{J_2} are algebraic integers, and divisible as algebraic integers by $q^{|J_2|}$. The lemma is proved.

By (3.10) and Lemma 3.1, we deduce that the eigenvalues of $q^{N+r-|J_1|-|J_2|}\alpha$ acting on B^{J_1,J_2} are divisible as algebraic integers by $q^{N+r-|J_1|}$. Since $|J_1| \leq N$, to prove Theorem 1.2b, it suffices to restrict to those factors in (3.10) with $J_1 = S_1$. Namely, Theorem 1.2b is reduced to proving the following result.

Lemma 3.2. *If the sequence $\{f_1, \cdots, f_r\}$ forms a regular sequence of $\mathbf{F}_q[x_1, \cdots, x_N]$, then the p-adic meromorphic function defined by*

$$F(X, T) = \prod_{J_2 \in S_2} \det(I - Tq^{r-|J_2|}\alpha|B^{S_1,J_2})^{(-1)^{r-|J_2|}} \tag{3.14}$$

is a p-adic entire function in T.

Proof. This lemma should be equivalent to a vanishing theorem on cohomology. Thus, we need to find a cohomological formula for $F(X, T)$. For $0 \leq l \leq r$, define

$$C_l(S) = \bigoplus_{J_2 \in S_2, |J_2|=l} B^{S_1, S_2 - J_2} e_{J_2}.$$

Then $C_l(S)$ is a subspace of $K_l(S)$, stable under the operator α. We use α_l again to denote the operator

$$\alpha_l = \bigoplus_{J_2 \in S_2, |J_2|=l} q^l \alpha,$$

acting on $C_l(S)$. This should not cause confusion since the space will be specified. Define the boundary operator $\partial_l : C_l(S) \to C_{l-1}(S)$ as follows: If $\eta \in B^{S_1, S_2-J_2} e_{J_2}$ with $J_2 = \{i_1, \cdots, i_l\}$ and $i_1 < \cdots < i_l$, then

$$\partial_l(\eta e_{J_2}) = \sum_{k=1}^{l} (-1)^{k-1} D_{i_k}(\eta) e_{J_2 - i_k},$$

where $D_{N+1}, \cdots, D_{N+r}$ are the second part of the differential operators $\{D_1, \cdots, D_{N+r}\}$ mentioned in (3.4). Note that D_j sends B^{S-j} to B^S. In this way, the operator $\alpha_.$ becomes a chain map of the complex $C.(S)$:

$$0 \to C_r(S) \to C_{r-1}(S) \to \cdots \to C_1(S) \to C_0(S) \to 0.$$

And we have the formula

$$\begin{aligned} F(X,T) &= \prod_{l=0}^{r} \det(I - T\alpha_l | C_l(S))^{(-1)^l} \\ &= \prod_{l=0}^{r} \det\left(I - T\alpha_l | H_l(C.(S))\right)^{(-1)^l}, \end{aligned}$$

where the H_l denote the homology groups of the complex $C_.(S)$. Lemma 3.2 is reduced to proving the following vanishing result.

Lemma 3.3. *If the sequence $\{f_1, \cdots, f_r\}$ forms a regular sequence of $\mathbf{F}_q[x_1, \cdots, x_N]$, then*

$$H_l(C_.(S)) = 0, \text{ for all } l > 0.$$

Proof. Let $B(\mathcal{O}_0)$ be the unit ball in B:

$$B(\mathcal{O}_0) = \{ \sum_{u \in \mathbf{Z}^{N+r} \cap C(\Delta)} A_u \pi^{w(u)} x^u | A_u \in \mathcal{O}_0, A_u \to 0 \text{ as } u \to \infty \}.$$

The unit ball $B(\mathcal{O}_0)$ is a flat, complete and separated $\mathcal{O}_0$-module. Sending an element $\sum A_u \pi^{w(u)} x^u$ to $\sum A_u x^u$ modulo π is a reduction homomorphism from $B(\mathcal{O}_0)$ into the polynomial ring $P = \mathbf{F}_q[x_1, \cdots, x_{N+r}]$. Since $C(\Delta)$ consists of all points in $\mathbf{R}^{N+r}$ with non-negative coordinates, the reduction homomorphism from $B(\mathcal{O}_0)$ to the polynomial ring P is surjective.

By working with $B(\mathcal{O}_0)$ instead of B, one finds in a standard manner [Mo, Theorem 8.5] that it suffices to prove that the reduction $\bar{C}_.(S)$ modulo π of the complex $C_.(S)$ is acyclic in positive dimensions. The reduction of the differential operator D_{N+i} is well known to be given by

$$\bar{D}_{N+i} = x_{N+i}\frac{\partial}{\partial x_{N+i}} + x_{N+i}f_i.$$

We want to identify the complex $\bar{C}_.(S)$ with a simpler Koszul complex. For this purpose, let

$$E_{N+i} = \frac{\partial}{\partial x_{N+i}} + f_i.$$

These are r commuting differential operators acting on the polynomial ring P as $\partial/\partial x_{N+i}$ kills f_j. Let $K_.(P,E)$ be the Koszul complex on P defined by the differential operators $E_{N+1}, \cdots, E_{N+r}$. Thus, for $0 \leq l \leq r$, we have

$$K_l(P,E) = \bigoplus_{|J_2|=l, J_2\in S_2} Pe_{J_2}.$$

Define a map from $K_l(P,E)$ to $\bar{C}_l(S)$ by the rule

$$\psi: \ \eta e_{J_2} \longrightarrow (\prod_{i\in S-J_2} x_i)\eta e_{J_2}.$$

It is easy to see that ψ defines an $\mathbf{F}_q$-vector space isomorphism between $K_l(P,E)$ and $\bar{C}_l(S)$, where

$$\bar{C}_l(S) = \bigoplus_{J_2\in S_2, |J_2|=l} P^{S_1, S_2-J_2}e_{J_2}$$

and P^J denotes the space of those polynomials in P which are divisible by all the x_j with $j \in J$.

One further checks that ψ is actually a chain map. This follows from the commutative diagram for $j \in J_2$:

$$\begin{array}{ccc} Pe_{J_2} & \stackrel{\psi}{\longrightarrow} & P^{S_1,S_2-J_2}e_{J_2} \\ \Big\downarrow E_j & & \Big\downarrow \bar{D}_j \\ Pe_{J_2-j} & \stackrel{\psi}{\longrightarrow} & P^{S_1,S_2-\{J_2-j\}}e_{J_2-j}. \end{array}$$

Thus, it suffices to prove that the Koszul complex $K_{.}(P,E)$ is acyclic in positive dimensions. This results from the following lemma.

Lemma 3.4. *If the sequence* $\{f_1, \cdots, f_r\}$ *forms a regular sequence of* $\mathbf{F}_q[x_1, \cdots, x_N]$, *then the sequence* $\{E_{N+1}, \cdots, E_{N+r}\}$ *forms a regular sequence for* $P = \mathbf{F}_q[x_1, \cdots, x_{N+r}]$.

Proof. Denote $\mathbf{F}_q[x_1, \cdots, x_N]$ by P_0. Then

$$P = P_0[x_{N+1}, \cdots, x_{N+r}].$$

We need to prove the claim that if $\phi_1, \cdots, \phi_r$ are in P such that $\sum_i E_{N+i}\phi_i = 0$, then there is a skew-symmetric set $a_{ij} \in P$ $(1 \leq i, j \leq r)$ such that $\phi_i = \sum_j E_{N+j}a_{ij}$. This is done by induction on the maximal degree $k = \max_i \deg(\phi_i)$, where $\deg(\phi_i)$ means the degree of ϕ_i in the last r variables $\{x_{N+1}, \cdots, x_{N+r}\}$. We now prove the claim and assume $\sum_i E_{N+i}\phi_i = 0$.

If $k \leq 0$ (the constant 0 has degree $-\infty$), then all ϕ_i are in the subring P_0. Since the differential operators $\partial/\partial x_{N+i}$ kill P_0, we have $\sum_i f_i\phi_i = \sum_i E_{N+i}\phi_i = 0$. By our assumption, the sequence $\{f_1, \cdots, f_r\}$ forms a regular sequence of P_0. Thus, there is a skew-symmetric set $a_{ij} \in P_0$ $(1 \leq i, j \leq r)$ such that $\phi_i = \sum_j f_j a_{ij} = \sum_j E_{N+j}a_{ij}$. The claim is proved for $k \leq 0$.

Assume now that k is a positive integer. Let $\phi_i^{(k)}$ be the degree k part of ϕ_i with respect to the variables $\{x_{N+1}, \cdots, x_{N+r}\}$. The differential operators $\partial/\partial x_{N+i}$ reduce the degree by 1 and the multiplication operators f_i keep the degree. Comparing the highest degree parts, one finds that $\sum_i f_i\phi_i^{(k)} = 0$. Since P is free over P_0, the sequence $\{f_1, \cdots, f_r\}$ also forms a regular sequence for the larger ring P. Thus, there is a skew-symmetric set $b_{ij} \in P$ $(1 \leq i, j \leq r)$ of degree k such that $\phi_i^{(k)} = \sum_j f_j b_{ij}$. Now,

$$\sum_{i=1}^{r} E_{N+i}\left(\phi_i - \sum_{j=1}^{r} E_{N+j}(b_{ij})\right) = 0$$

and for all $1 \leq i \leq r$,

$$\deg\left(\phi_i - \sum_{j=1}^{r} E_{N+j}(b_{ij})\right) < k.$$

By induction, the claim is proved.

4. The polar part of the Ax-Katz theorem

In this section, we prove a slight generalization of Theorem 1.3 by taking into account the terms actually appearing in the polynomials f_i. Recall that Δ_1 is the convex closure in $\mathbf{R}^{N+r}$ of the origin and the exponents of the non-zero terms in the polynomial g and $C(\Delta_1)$ is the cone generated by Δ_1. For $u \in C(\Delta_1)$, let $w(u)$ be the weight

$$w(u) = u_{N+1} + \cdots + u_{N+r}.$$

For $I \in S$, define

$$w_I(g) = \min\{w(u) \mid u \in \mathbf{Z}^{N+r} \cap C(\Delta_1),\ u_i > 0 \text{ for all } i \in I\}.$$

For $1 \leq i \leq N$, let

$$\nu_i = w_{S-i}(g).$$

By definition, it is clear that $\nu_i - r$ is a non negative integer for each $1 \leq i \leq N$ as $S_2 \subseteq S - i$.

Theorem 4.1. *Let X be a complete intersection defined by the vanishing of a regular sequence of polynomials $\{f_1, \cdots, f_r\}$ over $\mathbf{F}_q$ in N variables of degrees $d_1, \cdots, d_r$. Then, all the reciprocal poles β_j of $Z(X,T)^{(-1)^{N-r-1}}$ are divisible by $q^{\min_i(1+\nu_i-r)}$ as algebraic integers.*

Proof. We now restrict the operator α from B to B_1. Since the Fredholm determinants of α acting on both B and B_1 are the same, equation (3.10) and Lemmas (3.1)-(3.2) remain valid if we replace B by B_1. By (3.10) and Lemma 3.2, it suffices to prove that for all $J_1 \subset S_1$ ($J_1 \neq S_1$) and $J_2 \subseteq S_2$, the first non-trivial slope of the Newton polygon computed with respect to ord_q of the p-adic entire function defined by

$$Z(J_1, J_2) = \det(I - Tq^{N+r-|J_1|-|J_2|}\alpha|B_1^{J_1,J_2}), \tag{4.1}$$

is at least $\min_i(1+\nu_i)$. By corollary 4.3 in [AS1], this first non-trivial slope is at least

$$N + r - |J_1| - |J_2| + w_{J_1,J_2}(g). \tag{4.2}$$

By Lemma 4.5 in [AS1], the minimun of the numbers in (4.2) occurs when $|J_1| + |J_2|$ is as large as possible, namely, $J_2 = S_2$ and $J_1 = S_1 - i$ for some i since we assumed that $J_1 \neq S_1$. In the last case, the number in (4.2) reduces to $1+\nu_i$. By (3.10), this proves that all reciprocal poles β_j of $Z(X,T)^{(-1)^{N-r-1}}$ are divisible by $q^{\min_i(1+\nu_i-r)}$ as p-adic integers. This divisibility is actually in the stronger sense of algebraic integers as $\prod_j(1-\beta_j T)$, being the denominator of $Z(X,T)^{(-1)^{N-r-1}}$, is a polynomial with integer coefficients. One simply makes the change of variable $T \longrightarrow q^{-\min_i(\nu_i+1-r)}T$. A standard argument using various extension fields of $\mathbf{F}_q$ shows that the resulting denominator is still a polynomial with integer coefficients, see [Ka1]. Theorem 4.1 is proved.

Following [AS1, (5.7)], one checks that

$$\nu_j - r \geq \max\{0, \frac{(N-1-\sum_i d_i)}{\max_i d_i}\}.$$

Since $\nu_j - r$ is a *non-negative* integer for each $1 \leq j \leq N$, Theorem 4.1 implies Theorem 1.3.

Finally, we show that some of Ax's optimal hypersurface examples for his theorem are also optimal hypersurface examples for Theorem 1.3. Let $d \geq 3$ and write $N = bd + h$, where $0 < h \leq d$. We assume that d does not divide $(N-1)(N-2)$, i.e., $h > 2$. One checks that $\mu = b$ and $\nu = b$.

Let X be the hypersurface of degree d in $\mathbf{A}^N$ defined by the polynomial

$$f(x_1, \cdots, x_N) = \left(\sum_{i=0}^{b-1} x_{id+1}\cdots x_{(i+1)d}\right) + x_{bd+1}\cdots x_N.$$

Let $N(f)$ be the number of $\mathbf{F}_q$-rational points on the hypersurface X. It is shown in [Ax] that $q^b \| N(f)$. Using Ax's recursive formula for $N(f)$, it is straightforward to show more precisely that

$$N(f) = q^b(-1)^{bd+h-1}\left(1 - (bd+h-b)q + \cdots\right),$$

where the omitted terms are sums and differences of the terms of the form q^k with $k \geq 2$. This formula shows that

$$Z(X,T)^{(-1)^{bd+h-2}} = \frac{(1-q^b T)}{(1-q^{b+1}T)^{bd+h-b}} R(T),$$

where $R(T)$ is a rational function whose reciprocal zeroes and poles are divisible by q^{b+2}.

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