Mathematische Annalen manuscript No.
(will be inserted by the editor)

State spaces of JB^*-triples*

Matthew Neal · Bernard Russo

the date of receipt and acceptance should be inserted later –

Abstract. An atomic decomposition is proved for facially symmetric spaces which satisfy some affine geometric axioms compatible with notions from the quantum mechanical measuring process. This is then applied to yield, under appropriate assumptions, geometric characterizations, up to isometry, of the unit ball of the dual space of a JB^*-triple, and up to complete isometry, of one-sided ideals in C^*-algebras.

Key words. facially symmetric space – JB*-triple – Jordan decomposition – atomic decomposition – contractive projection – one-sided ideal – C*-algebra

Introduction

The Jordan algebra of self-adjoint elements of a C^*-algebra A has long been used as a model for the bounded observables of a quantum mechanical system, and the states of A as a model for the states of the system. The state space of this Jordan Banach algebra is the same as the state space of the C^*-algebra A and is a weak*-compact convex subset of the dual of A. With the development of the structure theory of C^*-algebras, and the representation theory of Jordan Banach algebras, the problem arose of determining which compact convex sets in locally convex spaces are affinely isomorphic to such a state space. In the context of ordered Banach spaces, such a characterization has been given for Jordan algebras in the pioneering paper by Alfsen and Shultz, [1].

After the publication of [1], and the corresponding result for C^*-algebras [4], there began in the 1980s a development of the theory of JB^*-triples which paralleled in many respects the functional analytic aspects of the theory of operator algebras. JB^*-triples, which are characterized by holomorphic properties of their unit ball, form a large class of Banach spaces supporting a ternary algebraic structure which includes C^*-algebras, Hilbert spaces, and spaces of rectangular matrices, to name a

Denison University, Granville, OH 43023, e-mail: nealm@denison.edu
University of California, Irvine, CA 92697-3875, e-mail: brusso@uci.edu

* Both authors are supported by NSF grant DMS-0101153

few examples. In particular, most of the axioms used by Alfsen and Shultz were shown to have non-ordered analogs in the context of JB^*-triples (see [12]). By the end of the decade, a framework was proposed by Friedman and Russo in [15] in which to study the analog of the Alfsen-Shultz result for JB^*-triples. A characterization of those convex sets which occur as the unit ball of the predual of an *irreducible* JB^*-triple was given in [18]. Since JB^*-triples have only a local order, the result characterizes the whole unit ball, which becomes the "state space" in this non-ordered setting.

Guided by the approach of Alfsen and Shultz in the binary context, it seemed clear that to prove a geometric characterization of predual unit balls of *global* (that is, not irreducible) JBW^*-triples would require a decomposition of the space into atomic and non-atomic summands and a version of spectral duality. These goals have remained elusive in the framework of the axioms used in [18]. In the present paper, by introducing the very natural axiom asserting the existence of a Jordan decomposition in the linear span of every norm-exposed face, we are able to prove the atomic decomposition. In addition, by imposing a spectral axiom every bit as justified as the one in the Alfsen-Shultz theory, we are able to give a geometric characterization of the unit ball of the dual of a global JB^*-triple. These results give positive answers to Problems 1,2 and 3 in [18]. Moreover, when combined with the recent characterization of ternary rings of operators (TROs) in terms of its linear matricial norm structure [23], we obtain a facial operator space characterization of TROs and one-sided ideals in C^*-algebras, which responds to a question of D. Blecher.

The main results of this paper are Theorems 4, 7, and 8, which we state here.

Theorem 2.2. *Let Z be a neutral strongly facially symmetric space satisfying the pure state properties, and satisfying JP and JD. Then $Z = Z_a \oplus^{\ell^1} N$, where Z_a and N are strongly facially symmetric spaces satisfying the same properties as Z, N has no extreme points in its unit ball, and Z_a is the norm closed complex span of the extreme points of its unit ball.*

Theorem 3.3. *The predual Z_* of a Banach space Z is isometric to a JB^*-triple if and only if Z is an L-embedded, base normed, strongly spectral, strongly facially symmetric space which satisfies the pure state properties and JP.*

Theorem 3.4. *Let A be a TRO. Then A is completely isometric to a left ideal in a C^*-algebra if and only if there exists a convex set $C = \{x_\lambda :$*

$\lambda \in \Lambda\} \subset A_1$ *such that the collection of faces*

$$F_\lambda := F_{\begin{bmatrix} 0 \\ x_\lambda/\|x_\lambda\| \end{bmatrix}} \subset M_{2,1}(A)^*,$$

form a directed set with respect to containment, $F := \sup_\lambda F_\lambda$ *exists, and* (a)-(d) *hold, where*

(a) The set $\{\begin{bmatrix} 0 \\ x_\lambda \end{bmatrix} : \lambda \in \Lambda\}$ *separates the points of* F*;*

(b) $F^\perp = 0$ (*that is, the partial isometry* $V \in (M_{2,1}(A))^{**}$ *with* $F = F_V$ *is maximal*)*;*

(c) $\langle F, \begin{bmatrix} 0 \\ x_\lambda \end{bmatrix}\rangle \geq 0$ *for all* $\lambda \in \Lambda$*;*

(d) $S_F^*\left(\begin{bmatrix} 0 \\ x_\lambda \end{bmatrix}\right) = \begin{bmatrix} 0 \\ x_\lambda \end{bmatrix}$ *for all* $\lambda \in \Lambda$.

This paper is organized as follows. In section 1 we recall the background on facially symmetric spaces and on JB^*-triples and TRO's. Section 2 is devoted to a proof of the atomic decomposition. The first subsection contains a result for certain contractive projections on facially symmetric spaces and the second subsection introduces and studies the Jordan decomposition property. The third subsection gives a geometric characterization of spin factors, improving on the main result of [17]. The main result of section 2, the atomic decomposition (Theorem 2.2), is proved in the fourth subsection.

The main applications occur in section 3. After giving some elementary results, interesting in their own right, on contractive projections on Banach spaces in the first subsection, the second subsection then uses all of the machinery developed up to there to give a geometric characterization of Cartan factors, improving on the main result of [18]. The spectral duality axiom is introduced in the next subsection and used together with the atomic decomposition to give a geometric characterization of the dual ball of a global JB^*-triple (Theorem 7). The final subsection applies the latter to give an operator space characterization of one-sided ideals in C^*-algebras (Theorem 8).

1. Preliminaries

Facially symmetric spaces were introduced in [14] and studied in [15] and [17]. In [18], the complete structure of *atomic* facially symmetric spaces was determined, solving a problem posed in [14]. It was shown, more precisely, that an irreducible neutral strongly facially symmetric space is linearly isometric to the predual of one of the Cartan factors of types 1 to 6, provided that it satisfies some natural and physically significant axioms, four in number, which are known to hold in the preduals of all JBW^*-triples. As in [1], the study of state spaces of Jordan algebras (see also the books [2],[3]), we shall refer to these axioms as the pure state properties. Since we can regard the entire unit ball of the dual of a JB*-triple as the "state space" of a physical system, cf. [14, Introduction], we have given a geometric characterization of such state spaces.

The project of classifying facially symmetric spaces was started in [17], where, using two of the pure state properties, denoted by STP and FE, geometric characterizations of complex Hilbert spaces and complex spin factors were given. The former is precisely a rank 1 JBW^*-triple and a special case of a Cartan factor of type 1, and the latter is the Cartan factor of type 4 and a special case of a JBW^*-triple of rank 2. For a description of all of the Cartan factors, see [8, pp. 292-3]. The explicit structure of a spin factor naturally embedded in a facially symmetric space was then used in [18] to construct abstract generating sets and complete the classification in the atomic case.

The main objective of this paper is to establish the global structure of facially symmetric spaces. While this is of interest in its own right, and for the purposes of physics, it also yields, when combined with the results of [23], a purely affine geometric operator space characterization of one sided ideals of C*-algebras.

1.1. Facially symmetric spaces

Let Z be a complex normed space. Elements $f, g \in Z$ are *orthogonal*, notation $f \perp g$, if $\|f+g\| = \|f-g\| = \|f\| + \|g\|$. A *norm exposed face* of the unit ball Z_1 of Z is a non-empty set (necessarily $\neq Z_1$) of the form $F_x = \{f \in Z_1 : f(x) = 1\}$, where $x \in Z^*, \|x\| = 1$. Recall that a *face* G of a convex set K is a non-empty convex subset of K such that if $g \in G$ and $h, k \in K$ satisfy $g = \lambda h + (1-\lambda)k$ for some $\lambda \in (0,1)$, then $h, k \in G$. In particular, an extreme point of K is a face of K. We denote the set of extreme points of K by $\operatorname{ext} K$. An element $u \in Z^*$ is called a *projective unit* if $\|u\| = 1$ and $\langle u, F_u^\perp \rangle = 0$. Here, for any subset S, $S^\perp$ denotes the set of all elements orthogonal to each element of S. $\mathcal{F}$ and $\mathcal{U}$ denote the collections of norm exposed faces of Z_1 and projective units in Z^*, respectively.

Motivated by measuring processes in quantum mechanics, we defined a *symmetric face* to be a norm exposed face F in Z_1 with the following property: there is a linear isometry S_F of Z onto Z, with $S_F^2 = I$ (we call such maps *symmetries*), such that the fixed point set of S_F is $(\overline{\mathrm{sp}} F) \oplus F^\perp$

(topological direct sum). A complex normed space Z is said to be *weakly facially symmetric* (WFS) if every norm exposed face in Z_1 is symmetric. For each symmetric face F we defined contractive projections $P_k(F),\ k = 0,1,2$ on Z as follows. First $P_1(F) = (I - S_F)/2$ is the projection on the -1 eigenspace of S_F. Next we define $P_2(F)$ and $P_0(F)$ as the projections of Z onto $\overline{\mathrm{sp}}F$ and $F^\perp$ respectively, so that $P_2(F) + P_0(F) = (I + S_F)/2$. A *geometric tripotent* is a projective unit $u \in \mathcal{U}$ with the property that $F := F_u$ is a symmetric face and $S_F^* u = u$ for some choice of symmetry S_F corresponding to F. The projections $P_k(F_u)$ are called *geometric Peirce projections.*

$\mathcal{GT}$ and $\mathcal{SF}$ denote the collections of geometric tripotents and symmetric faces respectively, and the map $\mathcal{GT} \ni u \mapsto F_u \in \mathcal{SF}$ is a bijection [15, Proposition 1.6]. For each geometric tripotent u in the dual of a WFS space Z, we shall denote the geometric Peirce projections by $P_k(u) = P_k(F_u), k = 0,1,2$. Also we let $U := Z^*, Z_k(u) = Z_k(F_u) := P_k(u)Z$ and $U_k(u) = U_k(F_u) := P_k(u)^*(U)$, so that we have the *geometric Peirce decompositions* $Z = Z_2(u) + Z_1(u) + Z_0(u)$ and $U = U_2(u) + U_1(u) + U_0(u)$. A symmetry corresponding to the symmetric face F_u will sometimes be denoted by S_u. Two geometric tripotents u_1 and u_2 are *orthogonal* if $u_1 \in U_0(u_2)$ (which implies $u_2 \in U_0(u_1)$) and *colinear* if $u_1 \in U_1(u_2)$ and $u_2 \in U_1(u_1)$. More generally, elements a and b of U are *orthogonal* if one of them belongs to $U_2(u)$ and the other to $U_0(u)$ for some geometric tripotent u. Two geometric tripotents u and v are said to be *compatible* if their associated geometric Peirce projections commute, *i.e.*, $[P_k(u), P_j(v)] = 0$ for $k, j \in \{0,1,2\}$. By [15, Theorem 3.3], this is the case if $u \in U_k(v)$ for some $k = 0,1,2$. For each $G \in \mathcal{F}$, v_G denotes the unique geometric tripotent with $F_{v_G} = G$.

A contractive projection Q on a normed space X is said to be *neutral* if for each $\xi \in X,\ \|Q\xi\| = \|\xi\|$ implies $Q\xi = \xi$. A normed space Z is *neutral* if for every symmetric face F, the projection $P_2(F)$ corresponding to some choice of symmetry S_F, is neutral.

A WFS space Z is *strongly facially symmetric* (SFS) if for every norm exposed face F in Z_1 and every $y \in Z^*$ with $\|y\| = 1$ and $F \subset F_y$, we have $S_F^* y = y$, where S_F denotes a symmetry associated with F.

The principal examples of neutral strongly facially symmetric spaces are preduals of JBW^*-triples, in particular, the preduals of von Neumann algebras, see [16]. In these cases, as shown in [16], geometric tripotents correspond to tripotents in a JBW^*-triple and to partial isometries in a von Neumann algebra. Moreover, because of the validity of the Jordan decomposition for hermitian functionals on JB*-algebras, $\mathrm{sp}_{\mathbf{C}} F$ is automatically norm closed (cf. Lemma 2.4).

In a neutral strongly facially symmetric space Z, every non-zero element has a *polar decomposition* [15, Theorem 4.3]: for $0 \neq f \in Z$ there exists a unique geometric tripotent $v = v(f) = v_f$ with $f(v) = \|f\|$ and $\langle v, \{f\}^\perp \rangle = 0$. Let $\mathcal{M}$ denote the collection of minimal geometric tripotents of U, *i.e.*, $\mathcal{M} = \{v \in \mathcal{GT} : U_2(v)$ is one dimensional$\}$. If Z is a neutral strongly SFS space satisfying PE, then the map $f \mapsto v(f)$ is a bijection of $\operatorname{ext} Z_1$ and $\mathcal{M}$ ([17, Prop. 2.4]).

A partial ordering can be defined on the set of geometric tripotents as follows: if $u, v \in \mathcal{GT}$, then $u \leq v$ if $F_u \subset F_v$, or equivalently ([15, Lemma 4.2]), $P_2(u)^* v = u$ or $v - u$ is either zero or a geometric tripotent orthogonal to u. Let $\mathcal{I}$ denote the collection of indecomposable geometric tripotents of U, *i.e.*, $\mathcal{I} = \{v \in \mathcal{GT} : u \in \mathcal{GT},\ u \leq v \Rightarrow u = v\}$. In general, $\mathcal{M} \subset \mathcal{I}$, and under certain conditions, (Proposition2.4(a) below and [17, Prop. 2.9]), $\mathcal{M}$ coincides with $\mathcal{I}$.

We now recall the definitions of the pure state properties and other axioms.

Definition 1.1. Let f and g be extreme points of the unit ball of a neutral SFS space Z. The *transition probability* of f and g is the number

$$\langle f|g \rangle := f(v(g)).$$

A neutral SFS space Z is said to satisfy *"symmetry of transition probabilities"* STP if for every pair of extreme points $f, g \in \operatorname{ext} Z_1$, we have

$$\overline{\langle f|g \rangle} = \langle g|f \rangle.$$

In order to guarantee a sufficient number of extreme points, the following definition was made in [17] and assumed in [18]. For the present paper, this definition is too strong and will be abandoned. It will turn out that the property (b) of Proposition 1.1 will be available to us and suffice for our purposes.

Definition 1.2. A normed space Z is said to be *atomic* if every symmetric face of Z_1 has an extreme point.

Definition 1.3. A neutral SFS space Z is said to satisfy property FE if every norm closed face of Z_1 different from Z_1 is a norm exposed face. We use the terminology PE for the special case of this that every extreme point of Z_1 is norm exposed.

The following consequence of atomicity will be more useful to us in this paper.

Proposition 1.1 ([17],Proposition 2.7). *If Z is an atomic SFS space satisfying PE, then*

(a) $U = \overline{sp}\,\mathcal{M}$ *(weak*-closure), where* $\mathcal{M}$ *is the set of minimal geometric tripotents.*
(b) $Z_1 = \overline{co}\,\mathrm{ext}\,Z_1$ *(norm closure).*

Definition 1.4. A neutral SFS space Z is said to satisfy the *"extreme rays property"* ERP if for every $u \in \mathcal{GT}$ and every $f \in \mathrm{ext}\,Z_1$, it follows that $P_2(u)f$ is a scalar multiple of some element in $\mathrm{ext}\,Z_1$. We also say that $P_2(u)$ preserves extreme rays.

Definition 1.5. A WFS space Z satisfies JP if for any pair u, v of orthogonal geometric tripotents, we have

$$S_u S_v = S_{u+v}, \tag{1.1}$$

where for any geometric tripotent w, S_w is the symmetry associated with the symmetric face F_w.

The property JP was defined and needed in [18] only for minimal geometric tripotents u and v. The more restricted definition given here is needed only in Proposition 2.4(b), where ironically, the involved geometric tripotents turn out to be minimal. (The assumption of JP is used in subsection 2.1 only for minimal geometric tripotents.) As in Remark 4.2 of [18],with identical proofs, JP implies the following important *joint Peirce* rules for orthogonal geometric tripotents u and v:

$$\begin{aligned}
Z_2(u+v) &= Z_2(u) + Z_2(v) + Z_1(u) \cap Z_1(v),\\
Z_1(u+v) &= Z_1(u) \cap Z_0(v) + Z_1(v) \cap Z_0(u),\\
Z_0(u+v) &= Z_0(u) \cap Z_0(v).
\end{aligned}$$

Definitions 1.1,1.3 and 1.4 are analogs of physically meaningful axioms in [1]. In the Hilbert space model for quantum mechanics, property JP for minimal geometric tripotents is interpreted as follows. Choose $\xi \otimes \xi$ to be the state exposed by a yes/no question v and $\eta \otimes \eta$ to be the state exposed by another u, and complete ξ, η to an orthonormal basis. For any state vector ζ expressed in this basis, the symmetry S_u (resp. S_v) changes the sign of the coefficient of ξ (resp. η) and S_{u+v} changes the sign of both coefficients.

We need the concept of L-embeddedness for the proofs of Theorems 3.2 and 3.3. This is defined as follows. A linear projection P on a Banach space X is called an *L-projection* if $\|x\| = \|Px\| + \|(I-P)x\|$ for every $x \in X$. The range of an L-projection is called an *L-summand.* The space X is said to be an *L-embedded space* if it is an L-summand in its second dual. These concepts are studied extensively in [19, Chapter IV]. The predual of a JB^*-triple is an example of an L-embedded space ([6]) and every L-embedded space is weakly sequentially complete ([19, Theorem 2.2,page 169].

The following is the main result of [18]. We have added the assumption of L-embeddedness, which seems to have been overlooked in [18]. This omission was discovered in the process of proving Theorem 3.2. More precisely, our Theorem 3.1 is needed in the proofs of [18, Lemmas 5.5 and 6.6]. In addition, our Proposition 2.1 is needed for [18, Theorem 3.12], and our Corollary 3.1 is needed three times in [18, Proposition 4.11]. Cartan factors are defined in the next subsection.

Theorem 1.1 ([18],Theorem 8.3). *Let Z be an atomic neutral strongly facially symmetric space satisfying FE, STP, ERP, and JP. If Z is L-embedded, then $Z = \oplus_\alpha^{\ell^1} J_\alpha$ where each J_α is isometric to the predual of a Cartan factor of one of the types 1-6. Thus Z^* is isometric to an atomic JBW^*-triple. If Z is irreducible, then Z^* is isometric to a Cartan factor.*

One of our main objectives in this paper is to be able to drop the assumption of atomicity in this result, i.e. to find a non-ordered analog of the main theorem of Alfsen-Shultz [1]. This will be achieved in our Theorem 3.3 below, but at the expense of some other axioms.

1.2. JB^*-triples and ternary rings of operators

A *Jordan triple system* is a complex vector space V with a *triple product* $\{\cdot,\cdot,\cdot\} : V \times V \times V \longrightarrow V$ which is symmetric and linear in the outer variables, conjugate linear in the middle variable and satisfies the Jordan triple identity

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}.$$

A complex Banach space A is called a *JB^*-triple* if it is a Jordan triple system such that for each $z \in A$, the linear map

$$D(z) : v \in A \mapsto \{z, z, v\} \in A$$

is Hermitian, that is, $\|e^{itD(z)}\| = 1$ for all $t \in \mathbf{R}$, with non-negative spectrum in the Banach algebra of operators generated by $D(z)$ and $\|D(z)\| = \|z\|^2$. A summary of the basic facts about JB*-triples can be found in [25] and some of the references therein, such as [22],[12], and [13].

A JB^*-triple A is called a *JBW^*-triple* if it is a dual Banach space, in which case its predual is unique, denoted by A_*, and the triple product is separately weak* continuous. The second dual A^{**} of a JB^*-triple is a JBW^*-triple.

The JB^*-triples form a large class of Banach spaces which include C^*-algebras, Hilbert spaces, spaces of rectangular matrices, and JB*-algebras. The triple product in a C*-algebra $\mathcal{A}$ is given by

$$\{x, y, z\} = \frac{1}{2}\,(xy^*z + zy^*x).$$

In a JB*-algebra with product $x \circ y$, the triple product is given by $\{x, y, z\} = (x \circ y^*) \circ z + z \circ (y^* \circ x) - (x \circ z) \circ y^*$. An element e in a JB*-triple A is called a *tripotent* if $\{e, e, e\} = e$ in which case the map $D(e) : A \longrightarrow A$ has eigenvalues 0, $\frac{1}{2}$ and 1, and we have the following decomposition in terms of eigenspaces

$$A = A_2(e) \oplus A_1(e) \oplus A_0(e)$$

which is called the *Peirce decomposition* of A. The $\frac{k}{2}$-eigenspace $A_k(e)$ is called the *Peirce k-space*. The *Peirce projections* from A onto the Peirce k-spaces are given by

$$P_2(e) = Q^2(e), \quad P_1(e) = 2(D(e) - Q^2(e)), \quad P_0(e) = I - 2D(e) + Q^2(e)$$

where $Q(e)z = \{e, z, e\}$ for $z \in A$. The Peirce projections are contractive.

For any tripotent v, the space $A_2(v)$ is a JB*-algebra under the product $x \cdot y = \{x \;\; v \;\; y\}$ and involution $x^\sharp = \{v \;\; x \;\; v\}$. JBW*-triples have an abundance of tripotents. In fact, given a JBW*-triple A and f in the predual A_*, there is a unique tripotent $v_f \in A$, called the *support tripotent* of f, such that $f \circ P_2(v_f) = f$ and the restriction $f|_{A_2(v_f)}$ is a *faithful positive* normal functional.

An important class of JBW*-triples are the following six types of *Cartan factors* (see [8, pp. 292-3]) :

type 1 $B(H, K)$, with triple product $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$,
type 2 $\{z \in B(H, H) : z^t = -z\}$,
type 3 $\{z \in B(H, H) : z^t = z\}$,
type 4 spin factor (defined below),
type 5 $M_{1,2}(\mathcal{O})$ with triple product $\{x, y, z\} = \frac{1}{2}(x(y^*z) + z(y^*x))$,
type 6 $M_3(\mathcal{O})$

where $\mathcal{O}$ denotes the 8 dimensional complex Octonians, $B(H, K)$ is the Banach space of bounded linear operators between complex Hilbert spaces H and K, and z^t is the transpose of z induced by a conjugation on H. Cartan factors of type 2 and 3 are obviously subtriples of $B(H, H)$, the latter notation is shortened to $B(H)$, while type 4 can be embedded as a subtriple of some $B(H)$. The type 3 and 4 are Jordan algebras with the usual Jordan product $x \circ y = \frac{1}{2}(xy + yx)$. Abstractly, a *spin factor* is a Banach space that is equipped with a complete inner product $\langle \cdot, \cdot \rangle$ and a conjugation j on the resulting Hilbert space, with triple product

$$\{x, y, z\} = \frac{1}{2}(\langle x, y \rangle z + \langle z, y \rangle x - \langle x, jz \rangle jy)$$

such that the given norm and the Hilbert space norm are equivalent.

An important example of a JB^*-triple is a *ternary ring of operators* (TRO). This is a subspace of $B(H)$ which is closed under the product xy^*z. Every TRO is (completely) isometric to a corner $pA(1-p)$ of a C*-algebra A. TRO's play an important role in the theory of quantized Banach spaces (operator spaces), see [11] for the general theory and [10] for the role of TRO's. For one thing, as shown by Ruan [24], the injectives in the category of operator spaces are TRO's (corners of injective C*-algebras) and not, in general, operator algebras.

Motivated by a characterization for JB*-triples as complex Banach spaces whose open unit ball is a bounded symmetric domain, we gave in [23] a holomorphic operator space characterization of TRO's up to complete isometry. As a consequence, we obtained a holomorphic operator space characterization of C*-algebras as well. Since a closed left ideal in a C*-algebra is a TRO, Theorem 1.2 below will allow us, in our facial operator space characterization of left ideals (Theorem 3.4) to restrict to TROs from the beginning. The following is the main result of [23].

Theorem 1.2 ([23],Theorem 4.3). *Let $A \subset B(H)$ be an operator space and suppose that $M_n(A)_0$ is a bounded symmetric domain for some $n \geq 2$. Then A is n-isometric to a ternary ring of operators (TRO). If $M_n(A)_0$ is a bounded symmetric domain for all $n \geq 2$, then A is ternary isomorphic and completely isometric to a TRO.*

2. Atomic decomposition of facially symmetric spaces

2.1. Contractive projections on facially symmetric spaces

In this subsection, we shall assume that Z is a strongly facially symmetric space with dual $U = Z^*$. If $\{v_i\}$ is a countable family of mutually orthogonal minimal geometric tripotents, then $v = \sup v_i$ exists as it is the support geometric tripotent of $\sum 2^{-i} f_i$, where $v_i = v_{f_i}$. This fact will be used in the proof of the following lemma.

Lemma 2.1. *Let $\{v_i\}$ be a countable family of mutually orthogonal minimal geometric tripotents, with $v =: \sup v_i$. Then $v = \sum_i v_i$ (w^*-limit).*

Proof. Note first that by [15, Cor. 3.4(a) and Lemma 1.8], for each $n \geq 1$,

$$\Pi_1^n(P_2(v_i) + P_0(v_i)) = \sum_1^n P_2(v_i) + P_0(\sum_1^n v_i),$$

so by [15, Cor. 3.4(b)],

$$P_2(\sum_1^n v_i)\Pi_1^n(P_2(v_i) + P_0(v_i)) = \sum_1^n P_2(v_i),$$

and hence $\sum_1^n P_2(v_i)$ is a contractive projection. For $\varphi \in Z$, by orthogonality,

$$\sum_1^n \|P_2(v_i)\varphi\| = \|\sum_1^n P_2(v_i)\varphi\| \leq \|\varphi\|,$$

so that $\sum_1^\infty \|P_2(v_j)\varphi\| \leq \|\varphi\|$ and with $Q_n := \sum_1^n P_2(v_i)$ and for $m \geq n$,

$$\|Q_m\varphi - Q_n\varphi\| = \|Q_n\varphi\| - \|Q_m\varphi\|$$

so that $Q_n\varphi$ converges to a limit, call it $Q\varphi$, and Q is a contractive projection.

For each $x \in U$, Q_n^*x converges in the weak*-topology to Q^*x. Applying this with $x = v$ and recalling that $P_2(v_i)^*v = v_i$, we obtain $\sum_1^n v_i = Q_n^*v \to y$ in the weak*-topology for some $y \in U_2(v)$. On the other hand, since $\langle y, \sum 2^{-i} f_i\rangle = 1$, by [15, Theorem 4.3(c)], we have $F_v \subset F_y$ and therefore by strong facial symmetry, $y = v + y_0$, where $y_0 \in U_0(v)$. Since $y \in U_2(v)$ we must have $y_0 = 0$ and hence $y = v$. $\square$

Lemma 2.2. *Suppose that Z is neutral and satisfies JP. Let $\{v_i\}$ be a countable family of mutually orthogonal minimal geometric tripotents, with $v =: \sup v_i$. Then $\cup_{i=1}^\infty [Z_2(v_i) \cup Z_1(v_i)]$ is norm total in $Z_2(v) + Z_1(v)$.*

Proof. Let W be the norm closure of the complex span of $\cup_{i=1}^\infty [Z_2(v_i) \cup Z_1(v_i)]$. We first show that

$$Z_2(v) + Z_1(v) \subset W. \tag{2.1}$$

If $\varphi \in Z_2(v) + Z_1(v)$ and $\varphi \notin W$, then there exists $x \in U$, $\|x\| \leq 1$ with $\langle x, \varphi\rangle \neq 0$ and $\langle x, W\rangle = 0$. We'll show that $x \in U_0(v)$. Since $\varphi \in Z_2(v) + Z_1(v)$ implies $\langle x, \varphi\rangle = 0$, this is a contradiction, proving (2.1).

Let $s_n = \sum_1^n v_j$ and for $\rho \in Z$ of norm one, let $\rho = \rho_2 + \rho_1 + \rho_0$ be its geometric Peirce decomposition with respect to s_n. By JP (for minimal geometric tripotents), $\rho_2, \rho_1 \in W$. Therefore

$$\langle s_n \pm x, \rho\rangle = \langle s_n, \rho_2\rangle \pm \langle x, \rho_0\rangle = \langle P_2(s_n)s_n \pm P_0(s_n)x, \rho\rangle$$

so that

$$|\langle s_n \pm x, \rho\rangle| \leq \|P_2(s_n)s_n \pm P_0(s_n)x\| = \max(\|P_2(s_n)s_n\|, \|P_0(s_n)x\|) = 1.$$

Thus $\|\sum_1^n v_i \pm x\| = 1$ so by Lemma 2.1, $\|v \pm x\| \leq 1$. By [15, Theorem 4.6], $v + U_0(v)_1$ is a face in the unit ball of U, and since $v = (v+x)/2 + (v-x)/2$, $v \pm x \in v + U_0(v)_1$, proving $x \in U_0(v)$ and hence (2.1).

To show that equality holds in (2.1), note first that it is obvious that $Z_2(v_i) \subset Z_2(v)$, and if $\varphi \in Z_1(v_i)$, then by compatibility, $P_0(v)\varphi = P_0(v)P_1(v_i)\varphi \in Z_1(v_i)$. But $P_0(v)\varphi = P_0(v)P_0(v_i)\varphi \in Z_0(v_i)$ so that $P_0(v)\varphi = 0$, as required. □

Corollary 2.1. $P_0(v) = \Pi_{i=1}^{\infty} P_0(v_i)$ *(strong limit).*

Proof. Let $Q_n = \Pi_1^n P_0(v_i)$. Let $\varphi \in Z$ have geometric Peirce decomposition $\varphi_2 + \varphi_1 + \varphi_0$ with respect to v. Since $Z_0(v) \subset Q_n(Z)$, $Q_n\varphi_0 = \varphi_0 \to \varphi_0 = P_0(v)\varphi$. It remains to show that $Q_n(\varphi_2 + \varphi_1) \to 0$. By Lemma 2.2, it suffices to prove that $Q_n\psi \to 0$ for every i and every $\psi \in Z_2(v_i) \cup Z_1(v_i)$. But for any $\psi \in Z_k(v_i)$ for $k = 2, 1$, $Q_n\psi = Q_nP_k(v_i)\psi = 0$ as soon as $n \geq i$. □

Proposition 2.1. *Suppose that Z is neutral and satisfies JP. Let $\{u_i\}_{i\in I}$ be an arbitrary family of mutually orthogonal minimal geometric tripotents. Then $Q := \Pi_{i\in I}P_0(u_i)$ exists as a strong limit and Q is a contractive projection with range $\cap_{i\in I}Z_0(u_i)$.*

Proof. Fix $f \in Z$. For each countable set $\lambda \subset I$, let $g_\lambda := \Pi_{i\in\lambda}P_0(u_i)f$, which exists as a norm limit by Corollary 2.1.

With $\alpha := \inf \|g_\lambda\|$, where λ runs over the countable subsets of I, we can find a sequence λ_n of countable sets such that $\alpha = \lim \|g_{\lambda_n}\|$, and hence a countable set $\mu = \cup_n \lambda_n \subset I$ such that $\|g_\mu\| = \alpha$. It remains to prove that

$$\Pi_{i\in I}P_0(u_i)f = \Pi_{i\in\mu}P_0(u_i)f.$$

For $\epsilon > 0$, choose a finite set $A_0 \subset \mu$ such that for all finite sets A with $A_0 \subset A \subset \mu$,

$$\|\Pi_{i\in A}P_0(u_i)f - \Pi_{i\in\mu}P_0(u_i)f\| < \epsilon.$$

By the neutrality of $P_0(u_j)$ and the definition of α, for any $j \notin \mu$,

$$P_0(u_j)\Pi_{i\in\mu}P_0(u_i)f = \Pi_{i\in\mu}P_0(u_i)f.$$

Hence, for any finite subset B with $A_0 \subset B \subset I$,

$$\begin{aligned}
&\|\Pi_{i\in B}P_0(u_i)f - \Pi_{i\in\mu}P_0(u_i)f\| \\
&= \|\Pi_{i\in B-\mu}P_0(u_i)[\Pi_{i\in B\cap\mu}P_0(u_i)f - \Pi_{i\in\mu}P_0(u_i)f]\| \\
&\leq \|\Pi_{i\in B\cap\mu}P_0(u_i)f - \Pi_{i\in\mu}P_0(u_i)f\| < \epsilon. \square
\end{aligned}$$

2.2. Jordan decomposition

In this subsection we introduce the Jordan decomposition property. We use it in place of atomicity to obtain Proposition 2.4, which contains the analogs of [17, Prop. 2.9] and [18, Prop. 2.4]. Propositions 2.2 and 2.3 and Lemmas 2.3 and 2.4 are taken from an unpublished note of Yaakov Friedman and the second named author in 1990.

Proposition 2.2. *Let F be a norm exposed face of the unit ball of a normed space Z, and let I denote the closed unit interval. The following are equivalent.*

(a) $(sp_{\mathbf{R}}F)_1 \subset co(IF \cup -IF)$.

(b) For each non-zero $f \in sp_{\mathbf{R}}F, \exists\ g,h \in \mathbf{R}^+F$ *with* $f = g - h$ *and* $\|f\| = \|g\| + \|h\|$.

(c) $\partial(sp_{\mathbf{R}}F)_1 \subset co(F \cup -F)$.

(d) For each non-zero $f \in sp_{\mathbf{R}}F, \exists\ g,h \in \mathbf{R}^+F$ *with* $f = g - h$ *and* $g \perp h$.

Proof. (a)$\Rightarrow$(c). If $f \in \mathrm{sp}_{\mathbf{R}}F$ and $\|f\| = 1$, then $f = \lambda\alpha\sigma - (1-\lambda)\beta\tau$, with $\alpha, \beta, \lambda \in I$ and $\sigma, \tau \in F$. If $\lambda = 0$ or 1, then $f \in \pm F$ so assume that $0 < \lambda < 1$. We have

$$1 = \|f\| = \|\lambda\alpha\sigma - (1-\lambda)\beta\tau\| \le \lambda\alpha + (1-\lambda)\beta \le \alpha \vee \beta \le 1.$$

Since $\lambda < 1,\ \alpha = \beta = 1$.

(c)$\Rightarrow$(b). If $0 \ne f \in sp_{\mathbf{R}}F$, then $\|f\|^{-1}f = \lambda\sigma - (1-\lambda)\tau$ with $\lambda \in I$ and $\sigma, \tau \in F$. Since $\|\lambda\sigma\| + \|(1-\lambda)\tau\| = \lambda + (1-\lambda) = 1$, we have

$$\|f\| = \|f\|(\|\lambda\sigma\| + \|(1-\lambda)\tau\|) = \|(\|f\|\lambda)\sigma\| + \|(\|f\|(1-\lambda))\tau\|.$$

(b)$\Rightarrow$(a). Let $f \in (sp_{\mathbf{R}}F)_1$ and assume $0 < \|f\| \le 1$. With $f = g - h$ and $\|f\| = \|g\| + \|h\|$ with $g, h \in \mathbf{R}^+F$, we have

$$f = \|g\|(\|g\|^{-1}g) + \|h\|(-\|h\|^{-1}h) + (1 - \|f\|) \cdot 0 \in co(IF \cup -IF).$$

(d)$\Rightarrow$(b). If $g \perp h$, then $\|f\| = \|g - h\| = \|g\| + \|h\|$.

(b)$\Rightarrow$(d). If $g, h \in \mathbf{R}^+F$ and $F = F_x$ for some $x \in U$ of norm one, then $\|g + h\| = g(x) + h(x) = \|g\| + \|h\|$. Therefore, $\|g \pm h\| = \|g\| + \|h\|$, *i.e.*, $g \perp h$. □

Definition 2.1. A norm exposed face of the unit ball of a normed space Z satisfies the *Jordan decomposition property* if (one of) the conditions of Proposition 2.2 holds.

It is elementary that if F satisfies the Jordan decomposition property, then $\mathrm{ext}\,(\mathrm{sp}_{\mathbf{R}}F)_1 = \mathrm{ext}\,F \cup \mathrm{ext}\,(-F)$.

Lemma 2.3. *Let Z be a neutral SFS space, let F be a norm exposed face of $Z_1$and let $f \in F$. Then $S_{v(f)}(F) \subset F$ and $P_2(f)F \subset \mathbf{R}^+F$, where $P_k(f)$ denotes $P_k(v(f))$ for $k = 0, 1, 2$.*

Proof. Let $F = F_u$ for some $u \in \mathcal{GT}$. Then by the minimality property of the polar decomposition ([15, Theorem 4.3(c)]), $F_{v(f)} \subset F_u$ and by strong facial symmetry, $S^*_{v(f)}u = u$. Thus if $\rho \in F_u$, $\langle S_{v(f)}\rho, u\rangle = \langle \rho, u\rangle = 1$ *i.e.*, $S_{v(f)}\rho \in F_u$, which proves the first statement.

By what was just proved,

$$[P_2(f) + P_0(f)](F) = \frac{I + S_{v(f)}}{2}(F) \subset F.$$

Thus, if $g \in F$,

$$\|P_2(f)g\| \left(\frac{P_2(f)g}{\|P_2(f)g\|}\right) + \|P_0(f)g\| \left(\frac{P_0(f)g}{\|P_0(f)g\|}\right) \in F.$$

Since F is a face, $P_2(f)g/\|P_2(f)g\| \in F$. □

Proposition 2.3. *In a neutral SFS space, the Jordan decomposition is unique whenever it exists,* i.e., *if $u \in \mathcal{GT}$ and for $i = 1, 2$, if $f = \sigma_1 - \tau_1 = \sigma_2 - \tau_2$ with $\tau_i, \sigma_i \in \mathbf{R}^+F_u$, $1 = \|f\| = \|\sigma_i\| + \|\tau_i\|$, then $\sigma_1 = \sigma_2$ and $\tau_1 = \tau_2$.*

Proof. Apply $P_2(\sigma_1)$ and $P_2(\tau_1)$ to $f = \sigma_1 - \tau_1 = \sigma_2 - \tau_2$ to obtain $\sigma_1 = P_2(\sigma_1)\sigma_2 - P_2(\sigma_1)\tau_2$ and $-\tau_1 = P_2(\tau_1)\sigma_2 - P_2(\tau_1)\tau_2$. Since $\sigma_i \perp \tau_i$,

$$\begin{aligned} 1 = \|\sigma_1\| + \|\tau_1\| &\le \|P_2(\sigma_1)\sigma_2\| + \|P_2(\sigma_1)\tau_2\| + \|P_2(\tau_1)\sigma_2\| + \|P_2(\tau_1)\tau_2\| \\ &= \|[P_2(\sigma_1) + P_2(\tau_1)]\sigma_2\| + \|[P_2(\sigma_1) + P_2(\tau_1)]\tau_2\| \\ &\le \|\sigma_2\| + \|\tau_2\| = 1. \end{aligned}$$

Therefore $\|\sigma_1\| = \|P_2(\sigma_1)\sigma_2\| + \|P_2(\sigma_1)\tau_2\|$.

Case 1. $P_2(\sigma_1)\tau_2 \neq 0$. In this case, $P_2(\sigma_1)\sigma_2 \neq 0$, otherwise we would have $\sigma_1 = 0$ and hence $\sigma_1 = \sigma_2 = 0$. We then have

$$\frac{\sigma_1}{\|\sigma_1\|} = \frac{\|P_2(\sigma_1)\sigma_2\|}{\|\sigma_1\|}\frac{P_2(\sigma_1)\sigma_2}{\|P_2(\sigma_1)\sigma_2\|} + \frac{\|P_2(\sigma_1)\tau_2\|}{\|\sigma_1\|}\left(\frac{-P_2(\sigma_1)\tau_2}{\|P_2(\sigma_1)\tau_2\|}\right).$$

Since F_u is a face,

$$\frac{-P_2(\sigma_1)\tau_2}{\|P_2(\sigma_1)\tau_2\|} \in F_u.$$

On the other hand, by Lemma 2.3, $P_2(\sigma_1)\tau_2 \in \mathbf{R}^+F_u$, so that $P_2(\sigma_1)\tau_2 = 0$, a contradiction, so this case does not occur.

Next, as above, apply $P_2(\sigma_2)$ and $P_2(\tau_2)$ to $f = \sigma_1 - \tau_1 = \sigma_2 - \tau_2$ to obtain $\sigma_2 = P_2(\sigma_2)\sigma_1 - P_2(\sigma_2)\tau_1$ and $-\tau_2 = P_2(\tau_2)\sigma_1 - P_2(\tau_2)\tau_1$. Since $\sigma_i \perp \tau_i$, as above we obtain $\|\sigma_2\| = \|P_2(\sigma_2)\sigma_1\| + \|P_2(\sigma_2)\tau_1\|$.

Case 2. $P_2(\sigma_2)\tau_1 \neq 0$. Exactly as in case 1, this implies that $P_2(\sigma_2)\sigma_1 \neq 0$ and leads to a contradiction unless $\sigma_2 = 0$. So this case does not occur.

Case 3. $P_2(\sigma_2)\tau_1 = P_2(\sigma_1)\tau_2 = 0$. In this case, $\sigma_1 = P_2(\sigma_1)\sigma_2$ and $\sigma_2 = P_2(\sigma_2)\sigma_1$, so that $\|\sigma_1\| = \|\sigma_2\|$. It follows that $\tau_2(v(\sigma_1)) = \langle P_2(\sigma_1)\tau_2, v(\sigma_1)\rangle = 0$ and

$$\begin{aligned}\|\sigma_1\| &= \sigma_1(v(\sigma_1)) = f(v(\sigma_1)) = \sigma_2(v(\sigma_1)) - \tau_2(v(\sigma_1))\\ &= \sigma_2(v(\sigma_1)) \leq \|\sigma_2\| = \|\sigma_1\|,\end{aligned}$$

implying $v(\sigma_2) \leq v(\sigma_1)$. Similarly, using $P_2(\sigma_2)\tau_1 = 0$ leads to $\|\sigma_2\| = \sigma_1(v(\sigma_2))$ and $v(\sigma_1) \leq v(\sigma_2)$.

Thus $v(\sigma_2) = v(\sigma_1)$, and we now have

$$\sigma_1 = P_2(\sigma_1)f = P_2(\sigma_1)\sigma_2 - P_2(\sigma_1)\tau_2 = P_2(\sigma_1)\sigma_2 = P_2(\sigma_2)\sigma_2 = \sigma_2. \square$$

Lemma 2.4. *Let F be a norm exposed face satisfying the Jordan decomposition property. Then*

(a) $sp_{\mathbf{R}}F \cap isp_{\mathbf{R}}F = \{0\}$.
(b) If Z is a neutral strongly symmetric space, then the projection of $sp_{\mathbf{C}}F = sp_{\mathbf{R}}F + isp_{\mathbf{R}}F$ onto $sp_{\mathbf{R}}F$ is contractive.

Proof. Let $h \in \mathrm{sp}_{\mathbf{R}}F \cap i\mathrm{sp}_{\mathbf{R}}F$, and suppose that $\|h\| = 1$. By Proposition 2.2,

$$h = \alpha if + \beta(-ig) = \gamma f_1 + \delta(-g_1)$$

for some $f, g, f_1, g_1 \in F$ and $\alpha, \beta, \gamma, \delta \in \mathbf{R}$ with $\alpha \geq 0, \beta = 1 - \alpha, \gamma \geq 0, \delta = 1 - \gamma, f \perp g$, and $f_1 \perp g_1$. With $F = F_u$ for some $u \in \mathcal{GT}$, we have $i(\alpha - \beta) = h(u) = \gamma - \delta$, so that

$$h = \frac{1}{2}i(f - g) = \frac{1}{2}(f_1 - g_1). \tag{2.2}$$

Applying successively $P_2(f)$ and $P_2(g)$ to (2.2) we obtain

$$if = P_2(f)f_1 - P_2(f)g_1 \text{ and } -ig = P_2(g)f_1 - P_2(g)g_1.$$

Therefore

$$\begin{aligned}2 = \|if\| + \|ig\| &\leq \|P_2(f)f_1\| + \|P_2(f)g_1\| + \|P_2(g)f_1\| + \|P_2(g)g_1\|\\ &= (\|P_2(f)f_1\| + \|P_2(g)f_1\|) + (\|P_2(f)g_1\| + \|P_2(g)g_1\|)\\ &\leq \|f_1\| + \|g_1\| = 2.\end{aligned}$$

If $P_2(f)f_1 = 0$, then $if = -P_2(f)f_1 \in \mathbf{R}^+F$, a contradiction. Similarly, $P_2(f)g_1 \neq 0$. Since iF is a face, and

$$if = \|P_2(f)f_1\| \left(\frac{P_2(f)f_1}{\|P_2(f)f_1\|}\right) + \|P_2(f)g_1\| \left(\frac{-P_2(f)g_1}{\|P_2(f)g_1\|}\right),$$

$$\frac{P_2(f)f_1}{\|P_2(f)f_1\|} \in iF.$$

On the other hand, by Lemma 2.3,

$$\frac{P_2(f)f_1}{\|P_2(f)f_1\|} \in F$$

also. This is a contradiction which proves (a).

Now let $g + ih \in \mathrm{sp}_{\mathbf{R}}F + i\mathrm{sp}_{\mathbf{R}}F$. Write $g = a\rho - b\sigma$ with $\rho \perp \sigma$, $\rho, \sigma \in F$, and $\|g\| = a + b$. Then $\langle g, v_\rho - v_\sigma\rangle = a + b$, and

$$\begin{aligned}\|g + ih\| &\geq |\langle g + ih, v_\rho - v_\sigma\rangle| = |a + b + i\langle h, v_\rho - v_\sigma\rangle| \\ &= [(a + b)^2 + \langle h, v_\rho - v_\sigma\rangle^2]^{1/2} \geq a + b = \|g\|,\end{aligned}$$

proving (b). □

If Z is a dual space, so that each norm exposed face is weak*-compact, then (b) and the Jordan decomposition property imply that $\mathrm{sp}_{\mathbf{C}}F$ is closed, so that $Z_2(F) = \mathrm{sp}_{\mathbf{C}}F$.

Definition 2.2. *A WFS space satisfies property JD if every symmetric face satisfies the Jordan decomposition property. In this case, we say that Z is* base normed.

It is important to note that this property is hereditary, that is, if Z satisfies JD, then so does any geometric Peirce space $Z_k(u)$. Indeed, if $F_w \cap Z_k(u)$ is a local face corresponding to a geometric tripotent $w \in U_k(u)$, and $\rho \in \mathrm{sp}_{\mathbf{R}}[F_w \cap Z_k(u)]$, then $\rho = \alpha g - \beta h$, with $g, h \in F_w$ and $\|\rho\| = \alpha + \beta$. From this it follows that $P_k(u)g, P_k(u)h \in F_w$ and $\rho = \alpha P_k(u)g - \beta P_k(u)h$.

Proposition 2.4. *Let Z be a SFS space satisfying JD.*

(a) $\mathcal{I}$=$\mathcal{M}$.
(b) Suppose furthermore that Z is neutral and satisfies JP. Let $v \in \mathcal{M}$ and suppose that $w \in \mathcal{GT}$ and $w \top v$. Then $w \in \mathcal{M}$.

Proof. Let $v \in \mathcal{I}$ and suppose F_v contains two distinct elements f_1, f_2 and set $f = f_1 - f_2$. Then $f = \alpha g - \beta h$ with $\alpha, \beta \in \mathbf{R}^+$ and $g, h \in F_v$. By evaluating at v one sees that $\alpha = \beta = 1/2$. Therefore F_v contains orthogonal elements g and h with orthogonal supports v_g and v_h such that $v_g \leq v$, $v_h \leq v$. Since $v \in \mathcal{I}$, $v_g = v = v_h$, a contradiction. Thus F_v consists of a single point and $v \in \mathcal{M}$. This proves (a).

To prove (b), we first show that $F_w \subset Z_1(v)$. Let $\psi = \psi_2 + \psi_1 + \psi_0$ be the Peirce decomposition of $\psi \in F_w \subset Z_2(w)$ with respect to v. We shall show that $\psi_2 = \psi_0 = 0$. In the first place, by [17, Prop. 2.4], $\psi_2 = P_2(v)\psi = \psi(v) f_v$ and since $v \in U_1(w)$, $f_v \in Z_1(w)$. (To see this last step, note that for $k = 0, 2$, $P_k(w) f_v = P_k(w) P_2(v) f_v = P_2(v) P_k(w) f_v = \langle P_k(w) f_v, v\rangle f_v = \langle f_v, P_k(w)^* v\rangle f_v = 0$.) On the other hand, since v and w are compatible, $\psi_2 = P_2(v)P_2(w)\psi = P_2(w)P_2(v)\psi \in Z_2(w)$, showing that $\psi_2 \in Z_2(w) \cap Z_1(w) = \{0\}$. Now $\psi = \psi_1 + \psi_0 \in F_w$, so $S_v\psi = -\psi_1 + \psi_0 \in -F_w$ by [15, Theorem 2.5], so that $\psi_0 \in \mathrm{sp}_{\mathbf{R}} F_w$. Hence, if $\psi_0 \neq 0$,we can write $\psi_0/\|\psi_0\| = \lambda\sigma - \mu\tau$ with $\sigma, \tau \in F_w$, $\lambda, \mu \geq 0$ and $\lambda + \mu = 1$. Since $\sigma, \tau \in F_w$, as shown above, $\sigma_2 = \tau_2 = 0$ and $\psi_0/\|\psi_0\| = \lambda(\sigma_1 + \sigma_0) - \mu(\tau_1 + \tau_0)$ so that $\lambda\sigma_1 - \mu\tau_1 = 0$, $\lambda = \mu = 1/2$ (since $\sigma_1(w) = \tau_1(w) = 1$) and $\|\sigma_0 - \tau_0\| = \|2\psi_0/\|\psi_0\|\| = 2$. Since

$$2 = \|\sigma_0 - \tau_0\| \leq \|\sigma_0\| + \|\tau_0\| \leq 1 + 1 = 2,$$

$\|\sigma_0\| = 1$ and by neutrality of $P_0(v)$ ([15, Lemma 2.1], $\sigma_1 = 0$, implying $\sigma_0 = \sigma \in F_w$, a contradiction as $\sigma_0(w) = \langle P_2(w)P_0(v)\sigma, w\rangle = \langle P_2(w)\sigma, P_0(v)^* w\rangle = 0$. Therefore $\psi_0 = 0$ and $F_w \subset Z_1(v)$.

Now that we know $F_w \subset Z_1(v)$, we show that F_w is a single point. Suppose to the contrary that there exist $g, h \in F_w$ with $g \neq h$. Then $f := g - h$ is a non-zero element of $\mathrm{sp}_{\mathbf{R}} F_w$, so $f = \sigma - \tau$ with $\sigma, \tau \in \mathbf{R}^+ F_w$ and $\|\sigma\| + \|\tau\| = \|f\|$. Since $\sigma(w) = \tau(w)$, $\sigma \neq 0$ and $\tau \neq 0$, and since $\sigma \perp \tau$, v_σ and v_τ are orthogonal geometric tripotents in $U_2(w)$ and hence $U_2(v_\sigma + v_\tau) \subset U_2(w)$. Moreover, by [15, Theorems 2.3,2.5],

$$\begin{aligned} U_2(w) &= \overline{\mathrm{sp}}^{w^*}\{v_G : G \in \mathcal{SF}, G \subset Z_2(w)\} \\ &\subset \overline{\mathrm{sp}}^{w^*}\{v_G : G \in \mathcal{SF}, G \subset Z_1(v)\} \\ &\subset \overline{\mathrm{sp}}^{w^*}\{v_G : G \in \mathcal{SF}, S_v(G) = -G\} = U_1(v). \end{aligned}$$

Then by [18, Cor. 2.3], $v \in U_1(v_\sigma) \cap U_1(v_\tau)$ and by JP, $v \in U_2(v_\sigma + v_\tau) \subset U_2(w)$, that is, $w \vdash v$, a contradiction. $\square$

2.3. Rank 2 faces; spin factor

In this section we assume that Z is a neutral, strongly facially symmetric space satisfying JD and JP.

Lemma 2.5. *Let $v \in \mathcal{M}$ and $\varphi \in Z_1(v)$, $\|\varphi\| = 1$, and suppose that $w := v_\varphi$ is minimal in $U_1(v)$. Then either φ is a global extreme point or the midpoint of two orthogonal global extreme points.*

Proof. Since w is minimal in $U_1(v)$, the face $F_w \cap Z_1(v)$ in $Z_1(v)$ exposed by w, considered as a geometric tripotent of $U_1(v)$, is the single point $\{\varphi\}$. For any element $\psi \in F_w$, $P_1(v)\psi = \varphi$, since for $k = 0,2$, $\psi_k(w) = \langle P_k(v)\psi, w\rangle = \langle \psi, P_k(v)^* w\rangle = 0$. Thus every $\psi \in F_w$ has the form $\psi = \psi_2 + \varphi + \psi_0$ where $\psi_k = P_k(v)\psi$ for $k = 0,2$.

If $F_w = \{\varphi\}$, there is nothing more to prove. So assume otherwise in the rest of this proof. As in the proof of Proposition 2.4(a), F_w then contains two orthogonal elements $\sigma = \sigma_2 + \varphi + \sigma_0$ and $\tau = \tau_2 + \varphi + \tau_0$. Further

$$\begin{aligned} 2 = \|\sigma - \tau\| &= \|\sigma_2 - \tau_2\| + \|\sigma_0 - \tau_0\| \leq \|\sigma_2\| + \|\tau_2\| + \|\sigma_0\| + \|\tau_0\| \\ &= \|\sigma_2 + \sigma_0\| + \|\tau_2 + \tau_0\| \leq \|\sigma\| + \|\tau\| = 2. \end{aligned}$$

This proves $\|\sigma_2 + \sigma_0\| = 1 = \|\tau_2 + \tau_0\|$, and setting $u := v_{\sigma_2} \in U_2(v)$ and $\tilde{u} = v_{\sigma_0} \in U_0(v)$ one obtains $\sigma(u + \tilde{u}) = \|\sigma_2\| + \|\sigma_0\| = \|\sigma_2 + \sigma_0\| = 1$ so $\sigma \in F_w \cap F_{u+\tilde{u}}$.

We show next that $F_w \cap F_{u+\tilde{u}}$ is the single point $\{\sigma\}$. Suppose to the contrary that $F_w \cap F_{u+\tilde{u}}$ is not a singleton. Then, as above, it contains two orthogonal elements σ' and τ' with $\|\sigma_2' - \tau_2'\| + \|\sigma_0' - \tau_0'\| = 2$.

We next claim that

$$u = v_{\sigma_2'} = v_{\tau_2'} \quad , \quad \tilde{u}(\sigma_0') = \|\sigma_0'\| \quad , \quad \tilde{u}(\tau_0') = \|\tau_0'\|.$$

Indeed,

$$1 = \langle u + \tilde{u}, \sigma_2' + \sigma_0'\rangle = \sigma_2'(u) + \sigma_0'(\tilde{u}) \leq |\sigma_2'(u)| + |\sigma_0'(\tilde{u})| \leq \|\sigma_2'\| + \|\sigma_0'\| \leq 1,$$

so that $\sigma_0'(\tilde{u}) = \|\sigma_0'\|$ and $\sigma_2'(u) = \|\sigma_2'\|$, and hence $v_{\sigma_2'} \leq u$, and $v_{\sigma_2'} = u$, since u, being a multiple of v, is minimal, and hence indecomposable. The proofs for τ_2' and τ_0' are similar.

We next show that there are positive numbers λ, μ and an extreme point ρ such that $\sigma_2' = \lambda\rho$ and $\tau_2' = \mu\rho$. Indeed, $\sigma_2' = P_2(v)\sigma' = \sigma'(v) f_v = \sigma_2'(v) f_v$, where f_v is the extreme point corresponding to $v \in \mathcal{M}$, and $\|\sigma_2'\| = \sigma_2'(u) = \sigma_2'(v) f_v(u)$. Since u is a multiple of v, $f_v(u) \neq 0$ and

$$\sigma_2' = \frac{\|\sigma_2'\|}{f_v(u)} f_v \text{ and similarly } \tau_2' = \frac{\|\tau_2'\|}{f_v(u)} f_v.$$

Writing $f_v(u) = re^{i\theta}$, we have $\sigma_2' = \frac{\|\sigma_2'\|}{r}(e^{-i\theta} f_v)$ and $\tau_2' = \frac{\|\tau_2'\|}{r}(e^{-i\theta} f_v)$.

Finally, assuming without loss of generality that $\lambda \geq \mu$, we have

$$2 = \|\sigma_2' - \tau_2'\| + \|\sigma_0' - \tau_0'\| = \|\sigma_2'\| - \|\tau_2'\| + \|\sigma_0' - \tau_0'\| \leq 1 + \|\tau_0'\| - \|\tau_2'\|,$$

which implies that $\tau_2' = 0$ and $\|\tau_0'\| = 1$. By neutrality of $P_0(v)$, $\tau' = \tau_0'$ which is a contradiction.

This proves that $F_w \cap F_{u+\tilde{u}}$ is a single point $\{\sigma\}$ and hence $\sigma = \sigma_2+\varphi+\sigma_0$ is a global extreme point. Then so is $\tilde{\sigma} := -S_v\sigma = -\sigma_2+\varphi-\sigma_0$ and $\varphi = (\sigma + \tilde{\sigma})/2$, completing the proof. □

We can now prove versions of [17, Prop. 3.2,Lemma 3.6] without assuming our space is atomic. First, we need the following lemma, the conclusion of which is in the hypotheses of [17, Prop. 3.2,Lemma 3.6].

Lemma 2.6. *Let $v \in \mathcal{M}$, and let $w \in \mathcal{GT} \cap U_1(v)$. Suppose that $w \notin \mathcal{M}$. Then F_w is a rank 2 face, that is, $w = w_1 + w_2$ where w_1 and w_2 are orthogonal minimal geometric tripotents.*

Proof. There are two possibilities: (i) w is minimal in $U_1(v)$; and (ii) w is not minimal in $U_1(v)$.

In case (i), w is the support geometric tripotent for some extreme point φ of the unit ball of $Z_1(v)$. Since by assumption φ is not a global extreme point, by Lemma 2.5, φ is the midpoint of two orthogonal global extreme points, and therefore w is the sum of two orthogonal minimal geometric tripotents.

In case (ii), $w = w_1 + w_2$ where $w_1, w_2 \in \mathcal{GT} \cap U_1(v)$, $w_1 \perp w_2$, and by [18, Cor. 2.3], $w_1 \top v$, $w_2 \top v$, so $w_1, w_2 \in \mathcal{M}$ by Proposition 2.4(b). □

Proposition 2.5. *Assume that Z also satisfies FE and STP. Let $v \in \mathcal{M}$, and let $w \in \mathcal{GT} \cap U_1(v)$. Suppose that $w \notin \mathcal{M}$. Then*

- *(a) If σ and τ are orthogonal elements of F_w, then σ and τ are extreme points, $\sigma + \tau = f_{w_1} + f_{w_2}$ and $v_\sigma + v_\tau = w$, where $w = w_1 + w_2$ according to Lemma 2.6.*
- *(b) Each norm exposed face of Z_1, properly contained in F_w, is a point.*
- *(c) If ρ is an extreme point of F_w, then there is a unique extreme point $\tilde{\rho}$ of F_w orthogonal to ρ.*
- *(d) With $\xi = (f_{w_1} + f_{w_2})/2$, $F_w = \cup\{[\xi, \rho] : \rho \in ext F_w\}$, where $[\xi, \rho]$ is the line segment connecting ξ and ρ.*

Proof. Case (i). w is minimal in $U_1(v)$.

(a) In the proof of Lemma 2.5, it was shown that if F_w contains two orthogonal elements, then these elements are global extreme points. Once this is known, the equalities $\sigma + \tau = f_{w_1} + f_{w_2}$ and $v_\sigma + v_\tau = w$ follow exactly as in the proof of [17, Prop. 3.2].

(b) Suppose that $F_u \subset F_w$ and $F_u \neq F_w$. By [14, Lemma 2.7] if $\sigma \in F_u$, there exists $\tau \in F_w$ with $\tau \perp \sigma$. Then σ and τ are extreme points. Thus F_u consists only of extreme points, and so it contains only one element.

(c) If ρ is an extreme point of F_w, then as in the proof of (b), there exists an extreme point $\tilde{\rho} \in F_w$ orthogonal to ρ. Since by (a), $\rho + \tilde{\rho} = f_{w_1} + f_{w_2}$, $\tilde{\rho}$ is unique.

(d) The proof is exactly the same as in [17, Lemma 3.6].

Case (ii). w is not minimal in $U_1(v)$.

In the first place, since $Z_1(v)$ satisfies JD, and $F_w \cap Z_1(v)$ is not a point, it must contain two orthogonal elements g and h with orthogonal supports v_g and v_h in $U_1(v)$. Then by [18, Cor. 2.3], $v_g \top v$, $v_h \top v$, so by Proposition 2.4(b), $v_g, v_h \in \mathcal{M}$ and g, h are global extreme points. After noting that $Z_1(v)$ satisfies FE and STP (by [15, Lemma 2.8,Cor. 4.12]), it now follows exactly as in the proof of case (i) that (a)-(d) hold for the face $F_w \cap Z_1(v)$. In particular $F_w \cap Z_1(v) =$

$\{\lambda\rho + (1-\lambda)\tilde{\rho} : \rho, \tilde{\rho} \in F_w \cap Z_1(v) \cap \operatorname{ext} Z_1, \rho \perp \tilde{\rho}, v_\rho + v_{\tilde{\rho}} = w, 0 \le \lambda \le 1\}.$

Now take two orthogonal elements $\sigma, \tau \in F_w$ and Peirce decompose each one with respect to v:

$$\sigma = \sigma_2 + \sigma_1 + \sigma_0 \quad , \quad \tau = \tau_2 + \tau_1 + \tau_0.$$

Since $\sigma_1, \tau_1 \in F_w$, as noted above we may write

$$\sigma_1 = \lambda\rho + (1-\lambda)\tilde{\rho} \quad , \quad \tau_1 = \mu\phi + (1-\mu)\tilde{\phi},$$

where ρ and $\tilde{\rho}$ are orthogonal global extreme points lying in $F_w \cap Z_1(v)$ with $v_\rho + v_{\tilde{\rho}} = w$, and similarly for $\phi, \tilde{\phi}$.

We can partially eliminate ϕ and $\tilde{\phi}$ as follows. Since $\tau_1 = P_1(v)\tau = P_2(w)P_1(v)\tau \in Z_2(w)$ and $w = v_\rho + v_{\tilde{\rho}}$, by [17, Lemma 2.3]

$$\tau_1 = c_1\rho + c_2\tilde{\rho} + \psi \tag{2.3}$$

for scalars c_1, c_2 and $\psi \in Z_1(\rho) \cap Z_1(\tilde{\rho})$. Since $|c_1| + |c_2| = \|c_1\rho + c_2\tilde{\rho}\| = \|P_2(v_\rho)\tau_1 + P_0(v_{\tilde{\rho}})\tau_1\| \le 1$ and since $1 = \tau_1(w) = c_1 + c_2 + \psi(w) = c_1 + c_2$ we have $c_1 + c_2 = 1$ and $0 \le c_1, c_2 \le 1$. Denote c_1 by c in what follows.

We shall now prove that

$$\tau_0, \sigma_0 \in Z_1(v_\rho) \text{ and } v_{\tau_0} v_{\sigma_0} \in \mathcal{M}, \tag{2.4}$$

and

$$\psi \text{ in (2.3) is zero.} \tag{2.5}$$

To prove (2.4), note that since $v_\rho \in U_1(v)$, v_ρ is compatible with v , so $P_k(v_\rho)\tau_0 \in Z_0(v)$ for $k = 2, 1, 0$. Since $\tau_2 = P_2(v)\tau = \langle \tau, v\rangle f_v$ and

$$\begin{aligned}
f_v &= P_2(\rho)f_v + P_1(\rho)f_v + P_0(\rho)f_v \\
&= \langle f_v, v_\rho\rangle\rho + P_1(\rho)f_v + P_0(\rho)P_2(v)f_v \\
&= \overline{\langle \rho, v\rangle}\rho + P_1(\rho)f_v + \langle P_0(\rho)f_v, v\rangle f_v \\
&= P_1(\rho)f_v,
\end{aligned}$$

it follows that $\tau_2 \in Z_1(\rho)$. Moreover, since $S^*_{v_\rho} w = w$, we have $S_{v_\rho} F_w \subset F_w$. Hence

$$S_{v_\rho}\tau = -\tau_2 + c\rho + (1-c)\tilde{\rho} - \psi + S_{v_\rho}\tau_0 \in F_w,$$

and therefore

$$\frac{\tau + S_{v_\rho}\tau}{2} = c\rho + (1-c)\tilde{\rho} + (\tau_0 + S_{v_\rho}\tau_0)/2 \in F_w.$$

Let $\tau' := (\tau_0 + S_{v_\rho}\tau_0)/2$. We'll show $\tau' = 0$. Recall that for any $\phi \in Z$,

$$\|P_1(v)\phi + P_0(v)\phi\| = \| - S_v[P_1(v)\phi + P_0(v)\phi]\| = \|P_1(v)\phi - P_0(v)\phi\|.$$

Hence, if $\tau' \neq 0$, then $c\rho + (1-c)\tilde{\rho} - (\tau_0 + S_{v_\rho}\tau_0)/2 \in F_w$, whence $\tau' \in \mathrm{sp}_{\mathbf{R}} F_w$, and by the property JD,

$$\tau'/\|\tau'\| = \alpha(\xi_1 + \xi_0) - (1-\alpha)(\eta_1 + \eta_0)$$

with $\xi, \eta \in F_w$ and $\alpha \in [0,1]$. Note here that

$$\xi_2 = P_2(v)P_2(w)\xi = \langle P_2(w)\xi, v\rangle f_v \in Z_1(w),$$

so $\xi_2 = 0$ and similarly $\eta_2 = 0$. As in Proposition 2.4(b), this implies $\alpha = 1/2, \xi_1 = \eta_1$, $\|\xi_0 - \eta_0\| = 2$ and $\|\xi_0\| = 1 = \|\eta_0\|$. By neutrality, $\xi_1 = 0 = \eta_1$, which contradicts the fact that $\xi = \xi_1 + \xi_0 \in F_w$. Thus, $\tau' = 0$, proving that $\tau_0 \in Z_1(v_\rho)$. A similar proof shows that $\sigma_0 \in Z_1(v_\rho)$.

Now $v_{\tau_0} \in U_0(v) \cap U_1(v_\rho)$, and if $v_{\tau_0} \vdash v_\rho$, then $v_\rho \in U_2(v_{\tau_0}) \subset U_0(v)$, by [15, Cor. 3.4], a contradiction. Now by the two case lemma ([18, Prop. 2.2]), $v_{\tau_0} \top v_\rho$ and v_{τ_0} is a minimal geometric tripotent by Proposition 2.4(b). A similar proof shows that $\sigma_0 \in \mathcal{M}$. This proves (2.4).

We next prove (2.5). Recall that $\tau = \tau_2 + c\rho + (1-c)\tilde{\rho} + \psi + \tau_0$, and note that

$$\tau' := -S_v S_{v_\rho}\tau = \tau_2 + c\rho + (1-c)\tilde{\rho} - \psi + \tau_0,$$

and $-S_v S_{v_\rho}\sigma = \sigma$. If we let $\tau'' := (\tau + \tau')/2$ then $\tau, \tau', \tau'' \in F_w \cap \sigma^\perp$, in particular $\psi = \tau'' - \tau' \in \sigma^\perp$. Suppose that ψ is not a multiple of a global extreme point. Since $\psi \in Z_1(v_\rho) \cap Z_1(v_{\tilde{\rho}})$, and v_ψ is not minimal, we have $v_\rho, v_{\tilde{\rho}} \in U_2(v_\psi)$ and $w = v_\rho + v_{\tilde{\rho}} \in U_2(v_\psi)$. But $v_\sigma \leq w \in U_2(v_\psi)$ and $v_\sigma \perp v_\psi$, implying $v_\sigma \in U_0(v_\psi) \cap U_2(v_\psi)$, a contradiction.

We conclude that $\psi = \alpha\varphi$ is a multiple of a global extreme point φ. From (2.3), if $\alpha \neq 0$, then φ is a difference of two elements of F_w, hence an extreme point of $(\mathrm{sp}_{\mathbf{R}} F_w)_1$, which implies that $\varphi \in F_w \cup F_{-w}$. This is a contradiction since $\pm\alpha = \alpha\varphi(w) = \psi(w) = \psi(v_\rho + v_{\tilde{\rho}}) = 0$. Hence $\alpha = 0$ proving (2.5).

We next show that $F_w \cap \{\sigma\}^\perp \cap \{\tau\}^\perp = \emptyset$. Suppose there exists a point τ' lying in $F_w \cap \{\sigma\}^\perp \cap \{\tau\}^\perp$. By the above calculations, one member of the set $\{\tau_1, (\tau')_1, \sigma_1\}$ is a convex combination of the other two. From this it follows exactly as in the proof of Lemma 2.5 that the corresponding convex combination of two elements of the orthogonal set $\{\tau, \tau', \sigma\}$ is an extreme point, which is a contradiction. Thus $F_w \cap \{\sigma\}^\perp \cap \{\tau\}^\perp = \emptyset$.

We can now complete the proof of (a), and (b)-(d) will follow as in case (i). If $F_{v_\tau} \neq \{\tau\}$, then by JD, F_{v_τ} contains two orthogonal elements g, h. But we have proved that in this case $F_w \cap \{g\}^\perp \cap \{h\}^\perp = \emptyset$. However, this set contains σ and this contradiction shows that τ (and by symmetry σ) is an extreme point. This completes the proof of Proposition 2.5. $\square$

Once we know the result of Proposition 2.5 above, the proof in [17] shows that the main result of [17] holds with atomic replaced by JD and JP. We formalize this in the next theorem.

Theorem 2.1. *Let Z be a neutral strongly facially symmetric space which satisfies FE, STP, JP and JD. If $v \in \mathcal{M}$ and $u \in \mathcal{GT} \cap U_1(v)$, then $Z_2(u)$ is isometric to the dual of a complex spin factor.*

Proof. The argument in [17], from [17, Corollary 3.7] to [17, Theorem 4.16] uses only the following results from [17] and does not otherwise invoke the atomic assumption made there: [17, Prop. 2.9,Cor. 2.11,Prop. 3.2,Lemma 3.6].

On the one hand, [17, Prop. 2.9] and [17, Cor. 2.11] remain true if atomic is replaced there by JD and JP, as shown in our Proposition 2.4(a). On the other hand, [17, Prop 3.2] remains true if atomic is replaced by JD and JP, as shown in our Proposition 2.5(a),(b),(c); and [17, Lemma 3.6] remains true if atomic is replaced by JD and JP, as shown in our Proposition 2.5(d). Thus Theorem 2.1 is proved. $\square$

2.4. Atomic decomposition

The following is the main result of this section.

Theorem 2.2. *Let Z be a neutral strongly facially symmetric space satisfying the pure state properties, and satisfying JP and JD. Then $Z = Z_a \oplus^{\ell^1} N$, where Z_a and N are strongly facially symmetric spaces satisfying the same properties as Z, N has no extreme points in its unit ball, and Z_a is the norm closed complex span of the extreme points of its unit ball.*

Proof. If Z has no extreme points in its unit ball, there is nothing to prove. If it has an extreme point, then there exists a maximal family $\{u_i\}_{i\in I}$ of mutually orthogonal minimal geometric tripotents. Let $Q := \Pi_{i\in I}P_0(u_i)$ be the contractive projection on Z with $Q(Z) = \cap_{i\in I}Z_0(u_i)$ guaranteed by Proposition 2.1. We shall show that $N := Q(Z)$ and $Z_a := (I-Q)(Z)$ have the required properties. By maximality, N has no extreme points in its unit ball.

For a finite subset A of I and $Q_A := \Pi_{i\in A}P_0(u_i)$, by JP,

$$\begin{aligned}(I - Q_A)(Z) & \qquad\qquad (2.6)\\ = Z_2(\Sigma_A u_i) \oplus Z_1(\Sigma_A u_i) &\\ = (\oplus_A Z_2(u_i)) \oplus (\oplus_{i\neq j}[Z_1(u_i)\cap Z_1(u_j)]) \oplus (\oplus_A[Z_1(u_i)\cap Z_0(\Sigma_{j\neq i}u_j)])\,. &\end{aligned}$$

Since $I-Q_A \to I-Q$ strongly, it follows that every element of $(I-Q)(Z)$ is the norm limit of elements from $\cup_A(I-Q_A)(Z)$. Since obviously $Z_2(u_i)\perp Q(Z)$, in order to prove $Z_a \perp N$, it suffices to prove that for every $i\in I$,

$$Z_1(u_i) \perp Q(Z). \qquad (2.7)$$

For each i, let $Q_i = \Pi_{j\in I-\{i\}}P_0(u_j)$ and for $\varphi\in Z_1(u_i)$, write $\varphi = Q_i\varphi + (I-Q_i)\varphi$. Note that

$$Q_i(Z_1(u_i)) = Z_1(u_i)\cap[\cap_{j\in I-\{i\}}Z_0(u_j)]$$

and that

$$(I-Q_i)(Z_1(u_i)) \text{ is the norm closure of } \oplus^{\text{finite}}_{j\in I-\{i\}}[Z_1(u_i)\cap Z_1(u_j)].$$

For the latter, note that for a finite subset $A\subset I-\{i\}$, if $Q_{i,A}$ denotes the partial product for Q_i, then

$$\begin{aligned}(I-Q_{i,A})P_1(u_i) &= \sum_A P_2(u_j)P_1(u_i) + \sum_{k\neq l}P_1(u_k)P_1(u_l)P_1(u_i)\\ &\quad + \sum_A P_1(u_j)P_0(\sum_{k\neq j}u_k)P_1(u_i)\\ &= 0+0+\sum_A P_1(u_j)P_1(u_i).\end{aligned}$$

Thus, $(I-Q_i)\varphi$ can be approximated in the norm by elements from spaces of the form $\oplus_{j\in A}[Z_1(u_i)\cap Z_1(u_j)]$, where A is a finite subset of $I-\{i\}$.

Now (2.7) is reduced to proving that $Q_i(Z_1(u_i))\perp Q(Z)$ and $(I-Q_i)(Z_1(u_i))\perp Q(Z)$. Since $Z_1(u_i)\cap Z_1(u_j)\subset Z_2(u_i+u_j)$ and $Q(Z)\subset Z_0(u_i+u_j)$, it is clear that $[Z_1(u_i)\cap Z_1(u_j)]\perp Q(Z)$. It remains to show that

$$\Big(Z_1(u_i)\cap[\cap_{j\in I-\{i\}}Z_0(u_j)]\Big)\perp Q(Z). \qquad (2.8)$$

Suppose $g \in Z_1(u_i) \cap [\cap_{j\in I-\{i\}} Z_0(u_j)]$ and $h \in Q(Z)$. Then either $v_g \vdash u_i$ or $v_g \top u_i$. In the first case, since by Theorem 2.1, $U_2(v_g)$ is isometric to a spin factor, there is a minimal geometric tripotent $\tilde{u}_i$ with $\tilde{u}_i \perp u_i$ and $\tilde{u}_i \in U_0(\Sigma_{j\in I-\{i\}} u_j)$. This contradicts the maximality. Therefore v_g is a minimal geometric tripotent and g is a multiple of an extreme point ψ. If $h = h_2 + h_1 + h_0$ is the geometric Peirce decomposition of h with respect to v_g, then since v_g is compatible with all the u_k, $h_j \in Q(Z)$. Now h_2 is also a multiple of ψ and $\psi \in Z_1(u_i)$; hence $h_2 \in Z_0(u_i) \cap Z_1(u_i) = \{0\}$. Since $v_{h_1} \in U_1(v_g)$, either $v_{h_1} \vdash v_g$ or $v_{h_1} \top v_g$. In the first case we would have $v_g \in U_2(v_{h_1}) \subset Q(Z)$, a contradiction. In the second case, h_1 would be a multiple of ψ, again a contradiction. We conclude that $h_1 = 0$ and therefore $h = h_0 \in Z_0(v_g)$ so that $g \perp h$ as required, proving (2.8) and thus the decomposition $Z = Z_a \oplus^{\ell^1} N$.

It is elementary that all the properties of Z transfer to any L-summand. Finally, the set of extreme points of the unit ball of Z which lie in $(I-Q)(Z)$ are norm total in $(I-Q)(Z)$, since every element from the right side of (2.6) is a linear combination of at most two extreme points by Lemma 2.5 and Proposition 2.5(d). □

3. Characterization of one-sided ideals in C^*-algebras

3.1. Contractive projections on Banach spaces

An interesting question about general Banach spaces, which is relevant to this paper, is to determine under what conditions the intersection of 1-complemented subspaces is itself 1-complemented. Although this may be true if the contractive projections onto the subspaces form a commuting family, we have been unable to prove it or find it in the literature, without adding some other assumptions. The hypothesis of weak sequential completeness used in Corollary 3.2 and Theorem 3.1 is satisfied in L-embedded spaces, as noted in subsection 1.1.

Lemma 3.1. *Let X be a Banach space and let $\{P_i\}_{i\in I}$ be a family of commuting contractive projections on X. Then $W := \cap_{i\in I} P_i^*(X^*)$ is the range of a contractive projection on X^*.*

Proof. Let $\mathcal{F}$ denote the collection of finite subsets of I. For each $A \in \mathcal{F}$, let $Q_A = \Pi_{i\in A} P_i$. Since the unit ball $B(X^*)_1$ is compact in the weak*-operator topology (= point-weak*-topology), there is a subnet $\{R_\delta\}_{\delta\in\mathcal{D}}$ of the net $\{Q_A^*\}_{A\in\mathcal{F}}$ converging in this topology to an element $R \in B(X^*)_1$. Thus $R_\delta = Q^*_{u(\delta)}$, where $u : \mathcal{D} \to \mathcal{F}$ is a finalizing map ($\forall A \in \mathcal{F}, \exists \delta_0 \in \mathcal{D}, u(\delta) \geq A, \forall \delta \geq \delta_0$), and for every $x \in X^*$ and $f \in X$,

$$\langle Rx, f\rangle = \lim_{\delta} \langle R_\delta x, f\rangle.$$

It is now elementary to show that $R^2 = R$ and $Rx = x$ if and only if $x \in W$. For completeness, we include the details.

For $x \in X^*,\ f \in X$,

$$\begin{aligned}\langle R^2x, f\rangle &= \lim_\delta \langle R_\delta Rx, f\rangle = \lim_\delta \langle Rx, Q_{u(\delta)}f\rangle \\ &= \lim_\delta \lim_{\delta'} \langle R_{\delta'}x, Q_{u(\delta)}f\rangle = \lim_\delta \lim_{\delta'} \langle x, Q_{u(\delta')}Q_{u(\delta)}f\rangle \\ &= \lim_{\delta'} \langle x, Q_{u(\delta')}f\rangle = \lim_{\delta'} \langle R_{\delta'}x, f\rangle = \langle Rx, f\rangle.\end{aligned}$$

Thus $R^2 = R$.

If $x \in W$, then $Q_A^* x = x$ for every $A \in \mathcal{F}$, so that $\langle Rx, f\rangle = \lim_\delta \langle R_\delta x, f\rangle = \lim_\delta \langle Q_{u(\delta)}^* x, f\rangle = \langle x, f\rangle$, so that $Rx = x$.

Conversely, if $Rx = x$, then

$$\begin{aligned}\langle P_i^* x, f\rangle &= \langle P_i^* Rx, f\rangle = \lim_\delta \langle R_\delta x, P_i f\rangle \\ &= \lim_\delta \langle P_i^* Q_{u(\delta)}^* x, f\rangle = \lim_\delta \langle R_\delta x, f\rangle = \langle Rx, f\rangle = \langle x, f\rangle,\end{aligned}$$

so that $x \in W$. □

We cannot conclude from the above proof that $\cap_{i\in I} P_i(X)$ is the range of a contractive projection on X. On the other hand, we have the following two immediate consequences.

Corollary 3.1. *Let X be a reflexive Banach space, and let $\{P_i\}_{i\in I}$ be a family of commuting contractive projections on X with ranges $X_i = P_i(X)$. Then $Y := \cap_{i\in I} X_i$ is the range of a contractive projection on X.*

Corollary 3.2. *Let X be a weakly sequentially complete Banach space, and let $\{P_i\}_{i\in \mathbf{N}}$ be a sequence of commuting contractive projections on X with ranges $X_i = P_i(X)$. Then $Y := \cap_{i\in \mathbf{N}} X_i$ is the range of a contractive projection on X.*

Proof. With $Q_n = P_1 \cdots P_n$, there is a subsequence $Q_{n_k}^*$ converging to an element $R \in B(X^*)_1$ in the weak*-operator topology, that is, for $x \in X^*$ and $f \in X$, $\langle x, Q_{n_k} f\rangle \to \langle Rx, f\rangle$, so that $\{Q_{n_k} f\}$ is a weakly Cauchy sequence. By assumption, $Q_{n_k} f$ converges weakly to an element Sf, and it is elementary to show that $R = S^*$, and S is a contractive projection on X with range Y. □

Theorem 3.1. *Let X be a weakly sequentially complete Banach space, and let $\{P_i\}_{i\in I}$ be a family of neutral commuting contractive projections on X with ranges $X_i = P_i(X)$. Then $Y := \cap_{i\in I} X_i$ is the range of a contractive projection on X.*

Proof. We note first that for any countable subset $\lambda \subset I$, by Corollary 3.2, there is a contractive projection Q_λ (not necessarily unique). with range $\cap_{i\in\lambda} X_i$ Now, for $f \in X$, define

$$\alpha_f = \inf_\lambda \inf_{Q_\lambda} \|Q_\lambda f\|.$$

There exists a sequence $\lambda^{(n)}$ and a choice of contractive projection $Q_{\lambda^{(n)}}$ with $\alpha_f \leq \|Q_{\lambda^{(n)}} f\| \leq \alpha_f + 1/n$. Set $\mu = \cup_n \lambda^{(n)}$ and let Q_μ be a contractive projection on X with range $\cap_{i\in\mu} X_i$. Since $Q_\mu(X) \subset Q_{\lambda^{(n)}}(X)$, we have $\|Q_\mu f\| = \|Q_\mu Q_{\lambda^{(n)}} f\| \leq \|Q_{\lambda^{(n)}} f\|$ implying $\alpha_f = \|Q_\mu f\|$, and so $\|Q_\mu f\| \leq \|Q_\lambda f\|$ for all countable subsets λ of I.

If Q'_μ is any other contractive projection with range $Q_\mu(X)$, then $Q_\mu f = Q'_\mu Q_\mu f = Q_\mu Q'_\mu f = Q'_\mu f$ so that we may unambiguously define an element $Qf \in \cap_{i\in\mu} X_i$ by $Qf := Q_\mu f$. By the neutrality of the projections, it follows that $Qf \in \cap_{i\in I} X_i$. Indeed, if $j \in I - \mu$, then

$$\|Q_{\mu\cup\{j\}} f\| = \|P_j Q_\mu f\| \leq \|Q_\mu f\| \leq \|Q_{\mu\cup\{j\}} f\|,$$

and by the neutrality of P_j, $P_j Q_\mu f = Q_\mu f$. Hence $Qf \in \cap_{i\in I} X_i$. Conversely, if $f \in \cap_{i\in I} X_i$, then in particular, $f \in Q_\mu(X)$, so $Qf = Q_\mu f = f$.

We have shown that Q is a nonlinear nonexpansive projection of X onto Y. It remains to show that Q is actually linear. For this it suffices to observe that, by neutrality, if $Qf = Q_\mu f$, then $Qf = Q_\lambda f$ for any countable set $\lambda \supset \mu$. Then, if $f, g \in X$ and $Qf = Q_\mu f$, $Qg = Q_\nu g$, and $Q(f+g) = Q_\sigma(f+g)$ for suitable countable sets μ, ν, σ of I, then with $\tau = \mu \cup \nu \cup \sigma$,

$$Q(f+g) = Q_\tau(f+g) = Q_\tau f + Q_\tau g = Qf + Qg.\square$$

3.2. Characterization of predual of Cartan factor

In this subsection we show that the entire machinery of [18] can be repeated with appropriate modifications to yield an extension of the main result of [18] to non-atomic facially symmetric spaces satisfying JD, and stated in Theorem 3.2 below. As noted below, the assumption that Z is L-embedded in its second dual needs to be added to the assumptions in [18]. As was done in the proof of Theorem 2.1, we shall explicitly indicate the modifications needed in [18], section by section, to prove Theorem 3.2.

In the proof of [18, Lemma 1.2] it was stated that the intersection of a certain family of 1-complemented subspaces, is itself 1-complemented. As noted in Section 3.1, this is problematical in general. However, [18, Lemma 1.2] is used in [18] only in the context of a reflexive Banach space, hence it is covered by Corollary 3.1. The role of the assumption of atomic in [18, Proposition 1.5] is to obtain the property expressed in Proposition 1.1(b). But as shown in Theorem 2.2, this property will be available to us.

By Proposition 2.4(b) and Lemma 2.6 respectively, [18, Proposition 2.4] and [18, Proposition 2.5] remain true with atomic replaced by JD. [18, Corollary 2.7] depends only on [18, Proposition 2.5] and Theorem 2.1, while the part of [18, Lemma 2.8] concerned with the property FE is immediate from [18, Corollary 2.7] and [18, Proposition 2.4]. Finally, [18, Corollary 2.9] is immediate from [17, Proposition 2.9] which, as already remarked in the proof of Theorem 2.1, remains true with atomic replaced by JD (Proposition 2.4(a)).

The only reliance on atomicity in [18, Section 3] occurs in [18, Lemma 3.2] and [18, Proposition 3.7]. The former depends only on [18, Corollaries 2.7 and 2.9] and the latter on [18, Proposition 1.5], which as just noted, are both valid with atomic replaced by JD. In the proof of [18, Proposition 3.12] it was stated that the intersection of a family of Peirce-0 subspaces of an orthogonal family of minimal geometric tripotents is 1-complemented, and in fact the net of partial products converges strongly to the projection on the intersection. As no proof was provided for this in [18], we provided a proof in Proposition 2.1. Recall that Proposition 2.1 was also the key engredient of the proof of the atomic decomposition in Theorem 2.2 above.

With these remarks we can now assert the following modification of [18, Theorem 3.14].

Proposition 3.1. *Let Z be a neutral SFS space and assume the pure state properties FE,ERP, and STP, and the properties JD and JP. Assume that there exists a minimal geometric tripotent v with $U_1(v)$ of rank 1 and a geometric tripotent u with $u \vdash v$. Then U has an M-summand which is linearly isometric with the complex JBW^*-triple of all symmetric "matrices" on a complex Hilbert space (Cartan factor of type 3). In particular, if Z is irreducible, then Z^* is isometric to a Cartan factor of type 3.*

The only possible reliance on atomicity in [18, Section 4] occurs in [18, Lemma 4.9] and [18, Proposition 4.11]. The former depends only on [17, Lemma 3.6], which is valid in the presence of JD by Proposition 2.5(d), and the latter on [18, Lemma 1.2], which as noted above is needed only for reflexive Banach spaces. But [18, Lemma 4.9] states explicitly that reflexivity. Note that the "classification scheme", embodied in [18, Proposition 4.20] does not involve atomic so is valid in the presence of JD.

The only reliance on atomicity in [18, Section 5] occurs in [18, Lemma 5.2], which depends on [17, Corollary 2.11]. As already noted, the latter is valid in the presence of JD. In the proof of [18, Lemma 5.5] it was stated that the intersection of a family of Peirce-0 subspaces of a family of geometric tripotents which are either orthogonal or collinear is 1-complemented, and in fact the net of partial products converges strongly to the projection on the intersection. As no proof was provided for this in [18], we provided a proof of the 1-complementedness of the intersection in Theorem 3.1. This is the only place in this paper and one of two places in [18] where the assumption of L-embeddedness is used. Although it is problematical whether the strong convergence of the partial products exists, nevertheless, it is sufficient to take a subnet of the net of partial sums in the proof of [18, Lemma 5.5]. The same remark applies to [18, Lemma 6.6].

With these remarks we can now assert the following modification of [18, Theorem 5.10].

Proposition 3.2. *Let Z be a neutral SFS space of spin degree 4, which is L-embedded and which satisfies FE,STP,ERP, and JP and JD. Then Z has an L-summand which is linearly isometric to the predual of a Cartan factor of type 1. In particular, if Z is irreducible, then Z^* is isometric to a Cartan factor of type 1.*

The only reliance on atomicity in [18, Section 6] occurs in [18, Lemma 6.2]. However, this dependence is on earlier results which have been established in the presence of JD. As noted above for [18, Lemma 5.5], [18, Lemma 6.6] holds under the assumption of L-embeddedness.

With these remarks we can now assert the following modification of [18, Theorem 6.8].

Proposition 3.3. *Let Z be a neutral SFS space of spin degree 6, which is L-embedded and which satisfies FE,STP,ERP, and JP and JD. Then Z has an L-summand which is linearly isometric to the predual of a Cartan factor of type 2. In particular, if Z is irreducible, then Z^* is isometric to a Cartan factor of type 2.*

The results of [18, Section 7] carry over verbatim in the presence of JD. The proof of [18, Theorem 7.1] on pages 75–79 of [18] yields the following modification.

Proposition 3.4. *Let Z be a neutral SFS space which satisfies FE, STP, ERP, and JP and JD, and let $v, \tilde{v}$ be orthogonal minimal geometric tripotents in $U := Z^*$ such that the dimension of $U_2(v + \tilde{v})$ is 8 and $U_1(v + \tilde{v}) \neq \{0\}$. Then there is an L-summand of Z which is isometric to the predual of a Cartan factor of type 5,* i.e., *the 16 dimensional JBW^*-triple of 1 by 2 matrices over the Octonions. In particular, if Z is irreducible, then Z^* is isometric to the Cartan factor of type 5.*

Similarly, the proof of [18, Theorem 7.8] appearing on pages 79–82 of [18] yields the following modification.

Proposition 3.5. *Let Z be a neutral SFS space of spin degree 10 which satisfies FE,STP,ERP, and JP and JD, and has no L-summand of type I_2. Then Z contains an L-summand which is isometric to the predual of a Cartan factor of type 6,* i.e., *the 27 dimensional JBW^*-triple of all 3 by 3 hermitian matrices over the Octonions. In particular, if Z is irreducible, then Z^* is isometric to the Cartan factor of type 6.*

Finally, the proof of [18, Theorem 8.2] on pages 83–84 of [18] yields the following modification.

Theorem 3.2. *Let Z be a neutral strongly facially symmetric space satisfying FE, STP, ERP, which is L-embedded and which satisfies JP and JD. For any minimal geometric tripotent v in U, there is an L-summand $J(v)$ of Z isometric to the predual of a Cartan factor of one of the types 1-6 such that $\hat{v} \in J(v)$. If Z is the norm closure of the complex linear span of its extreme points, then it is isometric to the predual of an atomic JBW^*-triple.*

3.3. Spectral duality and Characterization of dual ball of JB^*-triple

If Z is an L-embedded, base normed, neutral strongly facially symmetric space satisfying JP and the pure state properties, then by Theorems 3.2 and 2.2, its dual Z^* is a direct sum $Z^* = (Z_a)^* \oplus^{\ell^\infty} N^*$ where $(Z_a)^*$ is isometric to an atomic JBW^*-triple. We shall identify $(Z_a)^*$ with this JBW^*-triple in what follows.

Lemma 3.2. *Suppose that Z is as above and assume that Z is the dual of a Banach space B. For $a \in B$, if $\hat{a}$ denotes the canonical image of a in Z^*, and Q is the projection of Z^* onto $(Z_a)^*$, then $\|Q\hat{a}\| = \|a\|$.*

Proof. For $a \in B$ with $\|a\| = 1$, let g be an extreme point of the nonempty convex w*-compact set $\{f \in Z : \|f\| = 1 = f(a)\}$. Then $g \in \text{ ext } Z_1$, so g vanishes on N^*. Thus

$$1 = \|a\| = \|\hat{a}\| \geq \|Q\hat{a}\| \geq |\langle Q\hat{a}, g\rangle| = |\langle \hat{a}, g\rangle| = |\langle g, a\rangle| = 1.\square$$

In order to show that the space B is isometric to a JB^*-triple, it suffices to show that the image of the map $a \mapsto Q\hat{a}$ is closed under the cubing operation in $(Z_a)^*$, and is hence a subtriple of $(Z_a)^*$. To show this we need a spectral assumption on the elements of B. To make this definition, we need a lemma.

Lemma 3.3. *Let Z be a neutral WFS space satisfying PE. Let $\{F_B : B \in \mathcal{B}\}$ be a family of norm closed faces of Z_1, where $\mathcal{B}$ denotes the set of non-empty Borel subsets of the closed interval $[a, b]$.*

(a) Suppose that
(i) if $B_1 \cap B_2 = \emptyset$, then $F_{B_1} \perp F_{B_2}$ and $v_{B_1 \cup B_2} = v_{B_1} + v_{B_2}$.
For $f \in C[a, b]$, if $P = \{s_0, \ldots, s_n\}$ is a partition of $[a, b]$ and $T = \{t_1, \ldots, t_n\}$ are points with $s_{i-1} \leq t_i \leq s_i$, the Riemann sums $S(P, T, f) = \sum_1^n f(t_j) v_{(s_{j-1}, s_j]}$ converge in norm to an element $\int f\, dv_B = \int f(t)\, dv_B(t)$ of Z^* as the mesh $|P| = \min\{s_j - s_{j-1}\} \to 0$.
(b) Suppose that **(i)** *holds, with $[a, b] = [0, \|x\|]$ for some $x \in Z_*$, and suppose that x satisfies the further conditions:*
(ii) $\langle x, F_B \rangle \subset B$ for each interval $B \in \mathcal{B}$;
(iii) $S^*_{F_B} x = x$ for $B \in \mathcal{B}$;
(iv) $\langle x, F^{\perp}_{(0, \|x\|]} \rangle = 0$.
Then $x = \int t\, dv_B(t)$.

Proof. For the proof of (a), it suffices to show that for every $\epsilon > 0$, there is a $\delta > 0$, such that

$$\|S(P, T, f) - S(P', T', f)\| < \epsilon \text{ if } |P|, |P'| < \delta. \tag{3.1}$$

By the uniform continuity of f, let $\delta > 0$ correspondence to a tolerance of $\epsilon/2$. If $|P|, |P'| < \delta$, then $S(P, T, f) - S(P \cup P', T'', f)$, where T'' is any selection of points, is of the form $\sum_1^m \alpha_j v_j$, where $|\alpha_j| < \epsilon/2$ and $v_1, \ldots, v_m$ are orthogonal geometric tripotents. Thus

$$\|S(P, T, f) - S(P \cup P', T'', f)\| = \max_j |\alpha_j| < \epsilon/2$$

and (3.1) follows.

For the proof of (b), it suffices to prove that x is the weak*-limit of the Riemann sums corresponding to $f_0(t) := t$, for by (a), x will also be the norm limit. In what follows, $F_{(0, \|x\|]}$ will be denoted by F. By (iii) and (iv)

$$\langle x, Z_1(F) + Z_0(F) \rangle = 0.$$

Also, each Riemann sum $\sum t_j v_{(s_{j-1}, s_j]} \in U_2(F)$, so

$$\langle \sum t_j v_{(s_{j-1}, s_j]}, Z_1(F) + Z_0(F) \rangle = 0.$$

Since $Z_2(F) = \mathrm{sp}_{\mathbf{C}} F$, it suffices to prove that for every $\psi \in F$,

$$\langle x - S(P, T, f_0), \psi \rangle \to 0 \text{ as } |P| \to 0.$$

Since $v_F = \sum v_i$ where $v_i = v_{(s_{i-1},s_i]}$, if $\psi \in F \subset \oplus_i Z_2(v_i) \oplus \oplus_{i\neq j}[Z_1(v_i) \cap Z_1(v_j)]$, then

$$\begin{aligned}1 = \langle v_F, \psi\rangle &= \langle v_F, \sum P_2(v_i)\psi + \sum_{i\neq j} P_1(v_i)P_1(v_j)\psi\rangle \\ &= \sum \langle v_i, P_2(v_i)\psi\rangle \leq \sum \|P_2(v_i)\psi\| = \|\sum P_2(v_i)\psi\| \leq \|\psi\| = 1.\end{aligned}$$

Therefore

$$\begin{aligned}\psi &= \sum \|P_2(v_i)\| \frac{P_2(v_i)}{\|P_2(v_i)\|} + \sum_{i\neq j} P_1(v_i)P_1(v_j)\psi \\ &\in \mathrm{co}\,(F_{v_1} \cup \cdots \cup F_{v_n}) + \oplus_{i\neq j}[Z_1(v_i) \cap Z_1(v_j)].\end{aligned}$$

By (iii), $\langle x, Z_1(F_B)\rangle = 0$ for every $B \in \mathcal{B}$. Therefore $\langle x, \psi\rangle = \langle x, \sum \lambda_i \psi_i\rangle$, where $\psi_i \in F_{v_i}, \lambda_i \geq 0, \sum \lambda_i = 1$. Also, $\langle S(P,T,f_0), \psi\rangle = \langle \sum t_i v_i, \sum \lambda_j \psi_j\rangle = \sum t_i \lambda_i$.

By (ii), $\langle x, \psi_i\rangle \in (s_{i-1}, s_i]$, so

$$|\langle x - S(P,T,f_0), \psi\rangle| = |\sum \lambda_i(\langle x, \psi_i\rangle - t_i)| \leq |P|.$$

The lemma is proved. □

Let us observe that if Z is the dual of a JB^*-triple A, then each element $x \in A$ satisfies the conditions (i)-(iv) of Lemma 3.3. Indeed, if C denotes the JB*-subtriple of A generated by x, then C is isometric to a commutative C*-algebra and consists of norm limits of elements $p(x)$ where p is an odd polynomial on $(0, \|x\|]$, cf. [22, 1.15] and [5, p. 438]; and if W denotes the JBW^*-triple generated by x in A^{**}, then W is a commutative von Neumann algebra. Thus, if $x = w|x|$ is the polar decomposition of x in W, and $|x| = \int \lambda\, de_\lambda$ is the spectral decomposition of $|x|$ in W, and the face F_B is defined as the face exposed by the tripotent $we(B) \in A^{**}$, then the family $\{F_B : B \in \mathcal{B}\}$ satisfies (i), as shown in [20, Theorem 3.2]. It also follows from [20, Theorem 3.2] that for every $\epsilon > 0$, there is a partition of $[0, \|x\|]$ such that $\|x - \sum t_j v_{(s_{j-1},s_j]}\| < \epsilon$. If B is a subinterval of $[0, \|x\|]$, and $\rho \in F_B$, then with $v_j = v_{(s_{j-1},s_j]}$, $B_j = B \cap (s_{j-1}, s_j]$, there exist $\rho_k \in F_{B_k}$ (if $B_k \neq \emptyset$) and $\lambda_k \geq 0$ with $\sum \lambda_k = 1$ such that $\langle x, \rho\rangle$ is approximated by

$$\langle \sum t_j v_j, \sum_{B_k \neq \emptyset} \lambda_k \rho_k\rangle = \sum_{B_j \neq \emptyset} t_j \lambda_j \in \mathrm{co}\,(\cup_{B_j \neq \emptyset} B_j),$$

proving (ii). Again, using [20, Theorem 3.2] we shall show that (iii) and (iv) hold. Since x is approximated in norm by $\sum t_j v_j$, where $v_j = v_{(s_{j-1},s_j]}$, to prove (iii), it suffices to prove that $v_B v_B^* v_j v_B^* v_B = v_B v_B^* v_j = v_j v_B^* v_B$.

Since $v_B = \sum v_{B_j}$ where $B_j = B \cap (s_{j-1}, s_j]$, it is trivial to check that each of the terms $v_B v_B^* v_j v_B^* v_B$, $v_B v_B^* v_j$, $v_j v_B^* v_B$ collapses to v_{B_j}. Since the support of the spectral measure of $|x|$ lies in $[0, \|x\|]$, (iv) also holds.

There is another property of elements of a JB^*-triple that we need to incorporate into our definition. It is based on the following observation. If x is an element of a JB^*-triple A, let $f(x)$ denote the element of C which is the norm limit of odd polynomials p_n which converge uniformly to $f \in C_0([0, \|x\|)$, and let $\tilde{f}(x) = \int f(\lambda)\, de_\lambda$. Since $p_n(x) = \tilde{p}_n(x)$,

$$
\begin{aligned}
f(x) - \tilde{f}(x) = f(x) - p_n(x) + \tilde{p}_n(x) - \sum p_n(t_k) v_k \\
+ \sum p_n(t_k) v_k - \sum f(t_k) v_k + \sum f(t_k) v_k - \tilde{f}(x),
\end{aligned}
$$

which shows that $\tilde{f}(x) = f(x) \in A$.

Definition 3.1. A strongly facially symmetric space Z with a predual Z_* is *strongly spectral* if, for every element $x \in Z_*$, there exists a family $\{F_B : B \in \mathcal{B}\}$ of norm closed faces of the closed unit ball Z_1, where $\mathcal{B}$ is the set of nonempty Borel subsets of $(0, \|x\|]$, satisfying (i)-(iv) in Lemma 3.3 and which also satisfies

(v) For every $f \in C_0(0, \|x\|)$, the element $\int f\, dv_B$ is weak*-continuous, that is, lies in Z_*.

Although somewhat complicated, this condition is precisely the analogue of a strongly spectral compact base K of a base normed space V given by Alfsen and Shultz in [1]. There it is given simply as the condition that in the order unit space V_* each element a decomposes as an orthogonal difference $a_+ - a_-$ of two positive elements. Here orthogonal means that a_+ and a_- are supported on real spans of orthogonal faces of K. Since V_* is unital, the unit may be used together with a and this property to carve out an orthogonal collection of faces similar to the one above, and a lattice of orthogonal elements of V_* which generate a space which is isometric to a full space of continuous functions, and hence closed under the continuous functional calculus. Since there is no unit in our space Z_*, we must assume that elements $x \in Z_*$ may be decomposed in the above fashion, and that the resulting continuous functional calculus operates in Z_*. Note that this is entirely a linear property, and has obvious quantum mechanical significance. The faces F_B are the states corresponding to observations of some value in B for the observable x. The probability if this happening for a state ψ is $|\psi(v_B)|$.

We now have the following characterizations of JB*-triples. In this characterization, the property JP must hold for all orthogonal faces, not just extreme points. Thus it simply says that the (necessarily commutative) product of the symmetries S_F and S_G corresponding to orthogonal faces F and G is $S_{F\vee G}$.

Theorem 3.3. *The predual Z_* of a Banach space Z is isometric to a JB^*-triple if and only if Z is an L-embedded, base normed, strongly spectral, neutral strongly facially symmetric space which satisfies the pure state properties and JP.*

Before proving this theorem, we require one more lemma.

Lemma 3.4. *Let Ψ (resp. $\Psi^\perp$) denote the projection of Z onto its atomic part Z_a (resp. nonatomic part Z_n) given by Theorem 2.2. For any norm exposed face $G \subset Z_1$, $G_a := \Psi(G) \cap \partial Z_1$ and $G_n := \Psi^\perp(G) \cap \partial Z_1$ are faces in Z_a and Z_n respectively, and*

$$G = co\,(G_a \cup G_n). \tag{3.2}$$

Moreover, writing $G = F_w$ for some geometric tripotent w, then $\Psi^* w$ is a geometric tripotent, and

$$F_{\Psi^* w} = G_a. \tag{3.3}$$

Proof. To show that G_a is a face in $(Z_a)_1$, let $\lambda\rho + (1-\lambda)\sigma \in G_a$ where $\rho, \sigma \in (Z_a)_1$. Then $\lambda\rho + (1-\lambda)\sigma = \Psi f$ for some $f \in G$, and $f = \lambda\rho + (1-\lambda)\sigma + f_n$. Since $\|f\| = 1 = \|\lambda\rho + (1-\lambda)\sigma\|$, $f_n = 0$ and $\rho, \sigma \in G$, $\|\rho\| = 1 = \|\sigma\|$, and $\rho \in G \cap Z_a$, proving that G_a is a face. Similarly for G_n.

If $f \in G$ has decomposition $f = f_a + f_n = \|f_a\| \frac{f_a}{\|f_a\|} + \|f_n\| \frac{f_n}{\|f_n\|}$, then since G is a face, $\frac{f_a}{\|f_a\|}, \frac{f_n}{\|f_n\|} \in G$. This proves $\subset$ in (3.2). If $g_a := \Psi g \in \Psi G \cap \partial Z_1$ for some $g \in G$, then $\|g_a\| = 1$ so $g = g_a = \Psi g \in G$. A similar argument for $\Psi^\perp(G) \cap \partial Z_1$ proves $\supset$ in (3.2).

To prove (3.3), let $g \in \Psi(G) \cap \partial Z_1$. Then $\langle g, \Psi^* w\rangle = \langle g, w\rangle = 1$ so that $g \in F_{\Psi^* w}$. On the other hand, if $g \in F_{\Psi^* w}$, then $1 = \|g\| = \langle g, \Psi^* w\rangle = \langle \Psi g, w\rangle$ so that $\Psi g \in F_w$. Since $g = \Psi g + \Psi^\perp g$ and $\|g\| = \|\Psi g\|$, $\Psi^\perp g = 0$, $\Psi g = g$ and $g \in \Psi(G) \cap \partial Z_1$.

It remains to show that $\Psi^* w$ is a geometric tripotent, that is,

$$\langle \Psi^* w, (G_a)^\perp\rangle = 0.$$

Note first that $G^\perp = G_a^\perp \cap G_n^\perp$ by (3.2). If $\rho \in G_a^\perp$, $\langle \Psi^* w, \rho\rangle = \langle w, \Psi(\rho)\rangle$ and this will be zero if $\Psi(\rho) \in G^\perp$. To prove this, first let $\sigma \in G_a$. Then $\rho \perp \sigma$, hence $\Psi(\rho) \perp \Psi(\sigma)$ and since $\Psi(\sigma) = \sigma$, $\Psi(\rho) \in G_a^\perp$. Then $\Psi(\rho) \in G_a^\perp \cap G_n^\perp = G^\perp$ as required. □

Proof of Theorem 3.3. Assume that Z is a strongly facially symmetric space satisfying the hypotheses of the theorem. Suppose x is an element of Z_*. By the spectral axiom and Lemma 3.3, there is an element $y \in Z_*$ such that for $\epsilon > 0$ there exists $\delta > 0$ such that, with $f_0(t) = t$ and $f_1(t) = t^3$,

$$\|x - S(P,T,f_0)\| < \epsilon \text{ and } \|y - S(P',T',f_1)\| < \epsilon$$

for all partitions P, P' with mesh less than δ. Fix a common partition $P = \{s_0, \ldots, s_n\}$ with $|P| < \delta$, and write $v_i = v_{(s_{i-1},s_i]}$ and $(v_i)_a = \Psi^*(v_i)$. Then by (3.3),

$$\|\Psi^*(\hat{x}) - \sum t_i(v_i)_a\| < \epsilon \text{ and } \|\Psi^*(\hat{y}) - \sum t_i^3(v_i)_a\| < \epsilon.$$

Since in general $\|\{aaa\} - \{bbb\}\| \leq \|a-b\|(\|a\|^2 + \|a\|\|b\| + \|b\|^2)$, and since the $(v_i)_a$ are orthogonal tripotents in the JBW^*-triple $(Z_a)^*$, we have

$$\|\{\Psi^*(\hat{x}), \Psi^*(\hat{x}), \Psi^*(\hat{x})\} - \sum t_i^3(v_i)_a\| < 3\epsilon\|x\|^2,$$

and therefore $\|\{\Psi^*(\hat{x}), \Psi^*(\hat{x}), \Psi^*(\hat{x})\} - \Psi^*(y)\| < \epsilon(3\|x\|^2 + 1)$. It follows that $\Psi(\widehat{Z_*})$ is a norm closed subspace of the JBW*-triple Z_a^* that is closed under the cubing operation. Hence $\Psi(Z_*)$ is a subtriple of Z_a^* as required.

The converse, that the dual Z of a JB*-triple is a strongly facially symmetric space satisfying the conditions of the theorem, has already been mentioned above. That the spectral axiom is satisfied was shown preceding Definition 3.1. The proofs that it is a strongly facially symmetric base normed space can be found in [16], the proofs that it satisfies the pure state properties can be found in [12], the proof of the L-embeddedness can be found in [6], and the proof of FE can be found in [9]. □

We can restate Theorem 3.3 from another viewpoint as follows: for a Banach space X, its open unit ball is a bounded symmetric domain if and only if X^* is an L-embedded, base normed, strongly spectral, neutral strongly facially symmetric space which satisfies the pure state properties and JP.

3.4. One-sided ideals in C-algebras*

Together with Theorem 1.2, Proposition 3.7 and Theorem 3.4 below give facial and linear operator space characterizations of C*-algebras and left ideals of C*-algebras. This work was inspired by [7], in which Theorem 1.2 is used to characterize left ideals as TRO's which are simultaneously abstract operator algebras with right contractive approximate unit.

We start by motivating the main result of this subsection. Recall that a TRO is made into a JB^*-triple by symmetrizing the ternary product.

Remark 3.1. If J is a closed left ideal in a C*-algebra and J possesses a right identity e of norm 1, then J is a TRO and $E := \begin{bmatrix} 0 \\ e \end{bmatrix}$ is a maximal partial isometry in $M_{2,1}(J)$, that is, $P_0(E) = 0$.

Proof. By a remark of Blecher (see [7, Lemma 2.9]), $xe^* = x$ for all $x \in J$, so that $x = xe^*e$ and in particular, e is a partial isometry, and so is E.

For $\begin{bmatrix} x \\ y \end{bmatrix} \in M_{2,1}(J)$,

$$\begin{aligned} P_0(E)\begin{bmatrix} x \\ y \end{bmatrix} &= (I - EE^*)\begin{bmatrix} x \\ y \end{bmatrix}(I - E^*E) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 - ee^* \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}(I - e^*e) \\ &= \begin{bmatrix} x(1 - e^*e) \\ (1 - ee^*)y(1 - e^*e) \end{bmatrix} = 0. \square \end{aligned}$$

Conversely, we have the following.

Proposition 3.6. *Let A be a TRO. Suppose there is a norm one element x in A such that the face in $(M_{2,1}(A)^*)_1$ exposed by*

$$X := \begin{bmatrix} 0 \\ x \end{bmatrix} \in M_{2,1}(A)$$

is maximal. Then A is completely isometric to a left ideal in a C-algebra, which ideal contains a right identity element.*

Proof. Let $B = M_{2,1}(A)$. If V is the partial isometry in B^{**} such that $F_X = F_V$, then $X = V + P_0(V)^*X = V$, so that x is a partial isometry in A, which we denote by v.

We next prove that v is a right unitary in A; that is, $x = xv^*v$, for all $x \in A$. Indeed, for $x \in A$,

$$\begin{aligned} D(V,V)\begin{bmatrix} x \\ 0 \end{bmatrix} &= \frac{1}{2}\left(\begin{bmatrix} 0 \\ v \end{bmatrix}\begin{bmatrix} 0 \\ v \end{bmatrix}^*\begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} x \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ v \end{bmatrix}^*\begin{bmatrix} 0 \\ v \end{bmatrix}\right) \\ &= \begin{bmatrix} xv^*v/2 \\ 0 \end{bmatrix}, \end{aligned}$$

and

$$P_2(V)\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ v \end{bmatrix}\begin{bmatrix} 0 \\ v \end{bmatrix}^*\begin{bmatrix} x \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ v \end{bmatrix}^*\begin{bmatrix} 0 \\ v \end{bmatrix} = 0.$$

Since $P_1(V)\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$, and

$$\begin{bmatrix} xv^*v/2 \\ 0 \end{bmatrix} = D(V,V)\begin{bmatrix} x \\ 0 \end{bmatrix} = P_2(V)\begin{bmatrix} x \\ 0 \end{bmatrix} + \frac{1}{2}P_1(V)\begin{bmatrix} x \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x \\ 0 \end{bmatrix}.$$

We next show that the map $\psi : a \mapsto av^*$ is a complete isometry of A onto a closed left ideal J of the C*-algebra $\overline{AA^*}$ and vv^* is a right identity of J. In the first place, since $\|\psi(x)\|^2 = \|xv^*\|^2 = \|(xv^*)(xv^*)^*\| = \|xv^*vx^*\| = \|xx^*\|$, ψ is an isometry. By the same argument, with $W = \mathrm{diag}\,(v, v, \ldots, v)$, for $X \in M_n(A)$, $\|XW^*\| = \|X\|$, so that ψ is a complete isometry.

If $c \in \overline{AA^*}$ is of the form $c = ab^*$ with $a, b \in A$, and $y \in J := \psi(A)$, say $y = xv^*$, then $cy = ab^*xv^* \in Av^* = J$. By taking finite sums and then limits, J is a left ideal in C. Finally, with $e = vv^*$ and $y = xv^* \in J$, $ye = xv^*vv^* = xv^* = y$. □

For the general case we have the following result.

Theorem 3.4. *Let A be a TRO. Then A is completely isometric to a left ideal in a C*-algebra if and only if there exists a convex set $C = \{x_\lambda : \lambda \in \Lambda\} \subset A_1$ such that the collection of faces*

$$F_\lambda := F_{\begin{bmatrix} 0 \\ x_\lambda/\|x_\lambda\| \end{bmatrix}} \subset M_{2,1}(A)^*,$$

form a directed set with respect to containment, $F := \sup_\lambda F_\lambda$ exists, and (a)-(d) *hold, where*

(a) The set $\{\begin{bmatrix} 0 \\ x_\lambda \end{bmatrix} : \lambda \in \Lambda\}$ separates the points of F;

(b) $F^\perp = 0$ (that is, the partial isometry $V \in (M_{2,1}(A))^{**}$ with $F = F_V$ is maximal);

(c) $\langle F, \begin{bmatrix} 0 \\ x_\lambda \end{bmatrix}\rangle \geq 0$ for all $\lambda \in \Lambda$;

(d) $S_F^*\left(\begin{bmatrix} 0 \\ x_\lambda \end{bmatrix}\right) = \begin{bmatrix} 0 \\ x_\lambda \end{bmatrix}$ for all $\lambda \in \Lambda$.

Proof. We first assume that we have a closed left ideal L in a C*-algebra B. In this part of the proof, to avoid confusion with dual spaces, we denote the involution in B by $x^\sharp$. The set of positive elements of the open unit ball of the C*-algebra $L \cap L^\sharp$, which we will denote by $(u_\lambda)_{\lambda\in\Lambda}$, is a contractive right approximate unit for L. Let $u = \text{w}^*\text{-}\lim u_\lambda \in B^{**}$. Identifying L^{**} with $B^{**}u$, we now verify the properties (a)-(d).

For each λ, $u_\lambda/\|u_\lambda\| = v_\lambda + v_\lambda^0$ where $v_\lambda = \text{w}^*\text{-}\lim(u_\lambda/\|u_\lambda\|)^n$ is the support projection of $u_\lambda/\|u_\lambda\|$, that is, $F_{u_\lambda/\|u_\lambda\|} = F_{v_\lambda} \subset B^*$, and v_λ^0 is an element orthogonal to v_λ. Since $u_\lambda \uparrow u$, $u = \sup_\lambda r(u_\lambda/\|u_\lambda\|)$, where $r(u_\lambda/\|u_\lambda\|)$ is the range projection of $u_\lambda/\|u_\lambda\|$. For each fixed $\mu \in \Lambda$, we apply the functional calculus to $u_\mu/\|u_\mu\|$ as follows. Let $f_n(0) = 0$, $f_n(t) = 1$ on $[1/n, 1]$ and linear on $[0, 1/n]$. Then $f_n(u_\mu/\|u_\mu\|) \in (L \cap L^\sharp)_1^+$ and so as above $f_n(u_\mu/\|u_\mu\|) = v_{\lambda(\mu,n)} + v^0_{\lambda(\mu,n)}$ and $\sup_n v_{\lambda(\mu,n)} = r(u_\mu/\|u_\mu\|)$. Therefore

$$u = \sup_\mu r(u_\mu/\|u_\mu\|) = \sup_\mu \sup_n v_{\lambda(\mu,n)} \leq \sup_\lambda v_\lambda = v \text{ say.}$$

On the other hand, since $v_\lambda \leq (1 + \frac{1-\|u_\lambda\|}{\|u_\lambda\|})u$, it follows that $v \leq u$ and therefore $u = v$.

It is clear that

$$F_\lambda = F_{\begin{bmatrix} 0 \\ u_\lambda/\|u_\lambda\| \end{bmatrix}} = F_{\begin{bmatrix} 0 \\ v_\lambda \end{bmatrix}} \subset F_{\begin{bmatrix} 0 \\ u \end{bmatrix}},$$

and therefore that $\sup_\lambda F_\lambda$ exists. We show that it equals $F_{\begin{bmatrix} 0 \\ u \end{bmatrix}}$. Suppose that for some $a, b \in B^{**}$, $F_\lambda \subset F_{\begin{bmatrix} a \\ b \end{bmatrix}}$ for every λ. This is equivalent to

$$\begin{bmatrix} 0 \\ v_\lambda \end{bmatrix} = Q(\begin{bmatrix} 0 \\ v_\lambda \end{bmatrix}) \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ v_\lambda b^* v_\lambda \end{bmatrix},$$

or $v_\lambda b^* v_\lambda = v_\lambda$. On the other hand, since $v_\lambda^0 = u_\lambda/\|u_\lambda\| - v_\lambda \to 0$,

$$u_\lambda b^* u_\lambda = \|u_\lambda\|^2 (v_\lambda + v_\lambda^0) b^* (v_\lambda + v_\lambda^0) \to u,$$

so that $ub^*u = u$ and as above, $F_{\begin{bmatrix} 0 \\ u \end{bmatrix}} \subset F_{\begin{bmatrix} a \\ b \end{bmatrix}}$, proving that $\sup_\lambda F_\lambda = F_{\begin{bmatrix} 0 \\ u \end{bmatrix}}$.

Let us now prove (a). Since $u_\lambda \uparrow u$, the convergence is strong convergence. We claim first that $B \cap B_2^{**}(u)$ is weak*-dense in $B_2^{**}(u)$. Indeed with $x \in B_2^{**}(u)$ of norm 1, there is a net $b_\alpha \in B$ with $b_\alpha \to x$ strongly. Then $u_\lambda b_\alpha u_\mu \to uxu = x$, and since $u_\mu u = u_\mu r(u_\mu) u = u_\mu r(u_\mu) = u_\mu$, $u_\lambda b_\alpha u_\mu \in B \cap B_2^{**}(u)$, proving the claim. We claim next that $L \cap L^\sharp = B_2^{**}(u) \cap B$. If $y \in L \cap L^\sharp$, then $y = bu = uc^\sharp$ for some $b, c \in B^{**}$, hence $y = uyu \in B_2^{**} \cap B$. Since $B_2^{**}(u) = uB^{**}u \subset B^{**}u = L^{**}$, we have

$B_2^{**}(u) \cap B \subset L^{**} \cap B = L$. If $x \in B_2^{**}(u) \cap B$, then $x^\sharp \in B_2^{**}(u) \cap B$, proving that $x \in L \cap L^\sharp$.

Let M denote the TRO $\begin{bmatrix} L \\ L \end{bmatrix}$. Let f, g be two elements of $F_{\begin{bmatrix} 0 \\ u \end{bmatrix}}$ which are not separated by $\begin{bmatrix} 0 \\ C \end{bmatrix}$. It follows that $\begin{bmatrix} 0 \\ C \end{bmatrix}$ annihilates $f - g \in M_2^*(\begin{bmatrix} 0 \\ u \end{bmatrix})$. This contradicts the fact, implicit in the preceding paragraph, that the linear span of C is w*-dense in $L_2^{**}(u) = B_2^{**}(u)$. This proves (a).

To prove (b), it suffices to show that for $a, b \in L^{**}$,

$$\left(1 - \begin{bmatrix} 0 \\ u \end{bmatrix} \begin{bmatrix} 0 \\ u \end{bmatrix}^*\right) \begin{bmatrix} a \\ b \end{bmatrix} \left(1 - \begin{bmatrix} 0 \\ u \end{bmatrix}^* \begin{bmatrix} 0 \\ u \end{bmatrix}\right) = 0.$$

This reduces to

$$\begin{bmatrix} 1 & 0 \\ 0 & 1-u \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} (1-u) = \begin{bmatrix} a(1-u) \\ (1-u)b(1-u) \end{bmatrix} = 0,$$

which is true since u is a right identity for L^{**}.

To prove (c), let N denote the TRO $\begin{bmatrix} B \\ B \end{bmatrix}$. Note that $F_{\begin{bmatrix} 0 \\ u \end{bmatrix}}$ is the normal state space of $N_2^{**}\left(\begin{bmatrix} 0 \\ u \end{bmatrix}\right)$ and that

$$\begin{bmatrix} 0 \\ u_\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{u_\lambda} \end{bmatrix} [0, u] \begin{bmatrix} 0 \\ \sqrt{u_\lambda} \end{bmatrix}$$

is the square of the self-adjoint element

$$\begin{bmatrix} 0 \\ \sqrt{u_\lambda} \end{bmatrix}^\sharp = \begin{bmatrix} 0 \\ u \end{bmatrix} [0, \sqrt{u_\lambda}] \begin{bmatrix} 0 \\ u \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{u_\lambda} \end{bmatrix} \begin{bmatrix} 0 \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{u_\lambda} \end{bmatrix}.$$

Hence (c) follows.

From the proof of (c), $\begin{bmatrix} 0 \\ u_\lambda \end{bmatrix} \in N_2^{**}\left(\begin{bmatrix} 0 \\ u \end{bmatrix}\right)$, so it is fixed by S_F^*.

To prove the converse, assume that A is a TRO satisfying the conditions of the theorem. Let B denote the TRO $\begin{bmatrix} A \\ A \end{bmatrix}$. As in the first part of the proof, for each λ, there exists a partial isometry $v_\lambda \in A^{**}$ and an element $v_\lambda^0 \in A_0^{**}(v_\lambda)$ such that $\begin{bmatrix} o \\ x_\lambda/\|x_\lambda\| \end{bmatrix} = \begin{bmatrix} 0 \\ v_\lambda \end{bmatrix} + \begin{bmatrix} 0 \\ v_\lambda^0 \end{bmatrix}$ and

$F_{\begin{bmatrix} 0 \\ x_\lambda/\|x_\lambda\| \end{bmatrix}} = F_{\begin{bmatrix} 0 \\ v_\lambda \end{bmatrix}}$. Since $\sup_\lambda F_{\begin{bmatrix} 0 \\ v_\lambda \end{bmatrix}} = F$ exists, let $F = F_{\begin{bmatrix} u \\ v \end{bmatrix}}$ with $\begin{bmatrix} u \\ v \end{bmatrix}$ a partial isometry in $(M_{2,1}(A))^{**}$. We shall show that $u = 0$. In the first place, $P_2(\begin{bmatrix} 0 \\ v_\lambda \end{bmatrix})(\begin{bmatrix} u \\ v \end{bmatrix}) = \begin{bmatrix} 0 \\ v_\lambda \end{bmatrix}$, which reduces to $P_2(v_\lambda)v = v_\lambda$. Since $\begin{bmatrix} 0 \\ v \end{bmatrix}$ is the image of $\begin{bmatrix} u \\ v \end{bmatrix}$ under a contractive projection, $\|v\| \le 1$, and therefore $P_1(v_\lambda)v = 0$ (by [12, Lemma 1.5]). Thus $v = v_\lambda + v_\lambda^0$ with v_λ^0 orthogonal to v_λ, and it follows that the support partial isometry $u(v)$ of the element $v \in A^{**}$ satisfies $u(v) \ge v_\lambda$. It follows that $\begin{bmatrix} 0 \\ v_\lambda \end{bmatrix} \le \begin{bmatrix} 0 \\ u(v) \end{bmatrix}$ and since $\begin{bmatrix} u \\ v \end{bmatrix}$ is the least upper bound, we have $\begin{bmatrix} u \\ v \end{bmatrix} \le \begin{bmatrix} 0 \\ u(v) \end{bmatrix}$. Thus

$$\begin{bmatrix} u \\ v \end{bmatrix} = P_2(\begin{bmatrix} 0 \\ u(v) \end{bmatrix}) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ P_2(u(v))v \end{bmatrix} = \begin{bmatrix} 0 \\ v \end{bmatrix},$$

showing that $u = 0$.

Conditions (b) and (d) imply that $\begin{bmatrix} 0 \\ x_\lambda \end{bmatrix}$ lies in the von Neumann algebra $B_2^{**}(\begin{bmatrix} 0 \\ v \end{bmatrix})$ while condition (c) implies that $\begin{bmatrix} 0 \\ x_\lambda \end{bmatrix} \ge 0$ in that von Neumann algebra. In particular, $\begin{bmatrix} 0 \\ x_\lambda \end{bmatrix}$ is self-adjoint, $vx_\lambda^* v = x_\lambda$. We claim that condition (a) implies that $\begin{bmatrix} 0 \\ C \end{bmatrix}$ cannot annihilate any non-zero element of $B_2^*(\begin{bmatrix} 0 \\ v \end{bmatrix})$. Indeed, suppose $\begin{bmatrix} 0 \\ C \end{bmatrix}(\psi_1 - \psi_2) = 0$ where $\psi_1 - \psi_2$ is the Jordan decomposition of a functional ψ in the self adjoint part of $B_2^*(\begin{bmatrix} 0 \\ v \end{bmatrix})$. Note that since $\{v_\lambda\}$ is directed, and $v_\lambda \le x_\lambda \le v$, it follows that $\|\psi_1\| = \begin{bmatrix} 0 \\ v \end{bmatrix}(\psi_1) = \sup \begin{bmatrix} 0 \\ C \end{bmatrix}(\psi_1) == \sup \begin{bmatrix} 0 \\ C \end{bmatrix}(\psi_2) = \begin{bmatrix} 0 \\ v \end{bmatrix}(\psi_2) = \|\psi_2\|$ and this contradicts (a), as $\psi_1/\alpha, \psi_2/\alpha \in F$, where α is the common norm of ψ_1 and ψ_2. It follows that the bipolar $(\begin{bmatrix} 0 \\ C \end{bmatrix}_0)^0 = B_2^{**}(\begin{bmatrix} 0 \\ v \end{bmatrix})$. Consequently, the w* closure of $\mathrm{sp}_{\mathbf{C}} C$ is $A_2^{**}(v)$ and since the norm closure of a convex set is the same as its weak closure

$$A \cap A_2^{**}(v) = A \cap \overline{\mathrm{sp}\, C}^{\mathrm{w}^*} = A \cap \overline{\mathrm{sp}\, C}^{\|\cdot\|} = \overline{\mathrm{sp}\, C}^{\|\cdot\|}$$

is a C*-subalgebra of $A_2^{**}(v)$.

We are now in a position to show that A is completely isometric to a left ideal of a C*-algebra. Exactly as in the proof of the right unital case we have $A \subset Av^*v$. We define a map $\Psi : A \to AA^*$ by $\Psi(a) = av^*$. The crux of the matter is to show that the range of Ψ lies in AA^*. If that is the case, then since for $X, Y, Z \in M_n(A)$, with $D = \mathrm{diag}(v^*, \dots, v^*)$,

$$XY^*ZD = XD(YD)^*ZD,$$

ψ is a complete isometry. Moreover, if $b, c \in A$ then $(bc^*)av* = (bc^*a)v^*$ shows that the range of ψ is a left ideal. It remains to show that $Av^* \subset AA^*$.

Note first that, for $a \in A$, $av^*x_\lambda \in A$, since

$$\begin{aligned} av^*x_\lambda &= av^*x_\lambda^{1/2} \cdot x_\lambda^{1/2} = av^*(v(x_\lambda^{1/2})^*v)v^*x_\lambda^{1/2} \\ &= (av^*v)(x_\lambda^{1/2})^*(vv^*x_\lambda^{1/2}) = a(x_\lambda^{1/2})^*x_\lambda^{1/2} \in A. \end{aligned}$$

Next, since v belongs to the w*-closure of $\mathrm{sp}_{\mathbf{R}}\, C$, and for each $a \in A$, $\{av^*y : y \in \mathrm{sp}_{\mathbf{R}}\, C\}$ is a convex subset of A (since $av^*y = \sum \alpha_i av^*x_{\lambda_i} = \sum \alpha_i a(x_{\lambda_i}^{1/2})^*x_{\lambda_i}^{1/2} \in A$), it follows that a belongs to the norm closure of $\{av^*y : y \in \mathrm{sp}_{\mathbf{R}}\, C\}$. Now $va^*av^*y = va^*av^*vy^*v = va^*ay^*v = (ya^*a)^\sharp \in A \cap A_2^{**}(v)$ and therefore va^*a belongs to the norm closure of the set $\{va^*av^*y : y \in \mathrm{sp}_{\mathbf{R}}\, C\}$ and hence $va^*a \in A$. Using the triple functional calculus in the TRO A, we have

$$av^* = a^{1/3}(a^{1/3})^*a^{1/3}v^* = a^{1/3}(v(a^{1/3})^*a^{1/3})^* \in AA^*.\Box$$

In Theorem 3.4, the elements x_λ represent a right approximate unit cast in purely linear terms. Similar language can be used to characterize C*-algebras.

Proposition 3.7. *Let A be a TRO. Then A is completely isometric to a unital C*-algebra if and only if there is a norm one element x in A such that the complex linear span $sp_{\mathbf{C}}(F))$ of the face F in A^* exposed by x coincides with A^*.*

Note that a characterization of non-unital C*-algebras can also be given with obvious modifications as in Theorem 3.4.

From another viewpoint, we have characterized TRO's A up to complete isometry by *facial properties* of $M_n(A)^*$, since by Theorem 1.2, this is equivalent to finding an isometric characterization of JB*-triples U in terms of facial properties of U^*. This is exactly what we have done in Theorem 3.3, which is the non-ordered version of Alfsen-Shultz's facial characterization of state spaces of JB-algebras in the pioneering paper [1].

References

1. E. Alfsen and F. Shultz, *State spaces of Jordan algebras*, Acta Math. 140 (1978), 155–190.
2. E. Alfsen and F. Shultz, *State spaces of operator algebras*, Birkhäuser, 2001.
3. E. Alfsen and F. Shultz, *Geometry of state spaces of operator algebras*, Birkhäuser, 2003.
4. E. Alfsen, H. Hanche-Olsen and F. Shultz, *State spaces of C^*-algebras*, Acta Math. **144** (1980), 267–305.
5. J. Arazy and Wilhelm Kaup, *On continuous Peirce decompositions, Schur multipliers and the perturbation of triple functional calculus*, Math. Ann. **320** (2001), 431–461.
6. T. J. Barton and R. Timoney, *Weak*-continuity of Jordan triple products and its applications*, Math. Scand. **59** (1986), 177–191.
7. D. Blecher, *One-sided ideals and approximate identities in operator algebras*, J. Australian Math. Soc. (to appear)
8. T. Dang and Y. Friedman, *Classification of JBW^*-triple factors and applications*, Math. Scand. **61** (1987), 292–330.
9. C. M. Edwards and G. T. Ruttimann,*On the facial structure of the unit balls in a JBW^*-triple and its predual*, J. London Math . Soc. (2) **38**, (1988) 317-332
10. E. Effros, N. Ozawa and Z.-J. Ruan, *On injectivity and nuclearity for operator spaces*, Duke Math. J. **110** (2001), no. 3, 489–521.
11. E. Effros and Z. J. Ruan, *Operator Spaces*, Oxford University Press, 2000.
12. Y. Friedman and B. Russo, *Structure of the predual of a JBW^*-triple*, J. Reine Angew. Math. **356** (1985), 67–89.
13. Y. Friedman and B. Russo, *A Gelfand-Naimark theorem for JB^*-triples*, Duke Math. J. **53** (1986), 139–148.
14. Y. Friedman and B. Russo, *A geometric spectral theorem*, Quart. J. Math. Oxford (2) **37** (1986), 263–277.
15. Y. Friedman and B. Russo, *Affine structure of facially symmetric spaces*, Math. Proc. Cambridge Philos. Soc. **106** (1989), 107–124.
16. Y. Friedman and B. Russo, *Some affine geometric aspects of operator algebras*, Pac. J. Math. 137 (1989), 123–144.
17. Y. Friedman and B. Russo, *Geometry of the dual ball of the spin factor*, Proc. Lon. Math. Soc. **65** (1992), 142–174.
18. Y. Friedman and B. Russo, *Classification of atomic facially symmetric spaces*, Can. J. Math. **45** (1993), 33–87.
19. P. Harmand, D. Werner, and W. Werner, *M-ideals in Banach spaces and Banach algebras*, Lec. Notes in Math. 1547, Springer-Verlag 1993.
20. L. Harris, *A generalization of C^*-algebras*, Proc. Lon. Math. Soc. **42** (1981), 331–361.
21. B. Iochum and F. Shultz, *Normal state spaces of Jordan and von Neumann algebras*, J. Funct. Anal. 50 (1983), 317–328.
22. W. Kaup, *A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces*, Math. Zeit. **183** (1983), 503–529.
23. M. Neal and B. Russo *Operator space characterizations of C^*-algebras and ternary rings*, Pacific J. Math. **209** (2003), 339–364.
24. Z. -J. Ruan, *Injectivity of operator spaces*, Trans. Amer. Math. Soc. **315** (1989), 89–104.
25. B. Russo, *Structure of JB^*-triples*, In: Jordan Algebras, Proceedings of the Oberwolfach Conference 1992, Eds: W. Kaup,K. McCrimmon,H. Petersson, de Gruyter, Berlin (1994), 209–280.