DERIVATIONS

Introduction to non-associative algebra

OR

Playing havoc with the product rule?

PART III—MODULES AND DERIVATIONS

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HISTORY OF THESE LECTURES

PART I
ALGEBRAS
FEBRUARY 8, 2011

PART II
TRIPLE SYSTEMS
JULY 21, 2011

PART III
MODULES AND DERIVATIONS
FEBRUARY 28, 2012

PART IV
COHOMOLOGY
JULY 26, 2012

OUTLINE OF TODAY'S TALK

1. REVIEW OF PART I
ALGEBRAS
(FEBRUARY 8, 2011)

2. REVIEW OF PART II
TRIPLE SYSTEMS
(JULY 21, 2011)

3. MODULES

4. DERIVATIONS INTO A MODULE

PRE-HISTORY OF THESE LECTURES

THE RIEMANN HYPOTHESIS

PART I
PRIME NUMBER THEOREM
JULY 29, 2010

PART II
THE RIEMANN HYPOTHESIS
SEPTEMBER 14, 2010

WHAT IS A MODULE?

The American Heritage Dictionary of the English Language, Fourth Edition 2009.

HAS 8 DEFINITIONS

1. A standard or unit of measurement.
2. **Architecture** The dimensions of a structural component, such as the base of a column, used as a unit of measurement or standard for determining the proportions of the rest of the construction.
3. **Visual Arts/Furniture** A standardized, often interchangeable component of a system or construction that is designed for easy assembly or flexible use: a sofa consisting of two end modules.
4. **Electronics** A self-contained assembly of electronic components and circuitry, such as a stage in a computer, that is installed as a unit.

5. **Computer Science** A portion of a program that carries out a specific function and may be used alone or combined with other modules of the same program.
6. **Astronautics** A self-contained unit of a spacecraft that performs a specific task or class of tasks in support of the major function of the craft.
7. **Education** A unit of education or instruction with a relatively low student-to-teacher ratio, in which a single topic or a small section of a broad topic is studied for a given period of time.
8. **Mathematics** A system with scalars coming from a ring.

1. REVIEW OF PART I—ALGEBRAS

AXIOMATIC APPROACH

AN ALGEBRA IS DEFINED TO BE A SET (ACTUALLY A VECTOR SPACE) WITH TWO BINARY OPERATIONS, CALLED ADDITION AND MULTIPLICATION

ACTUALLY, IF YOU FORGET ABOUT THE VECTOR SPACE, THIS DEFINES A

RING

ADDITION IS DENOTED BY

$a + b$

AND IS REQUIRED TO BE COMMUTATIVE AND ASSOCIATIVE

$$a + b = b + a, \quad (a + b) + c = a + (b + c)$$

THERE IS ALSO AN ELEMENT 0 WITH THE PROPERTY THAT FOR EACH a,

$$a + 0 = a$$

AND THERE IS AN ELEMENT CALLED $-a$ SUCH THAT

$$a + (-a) = 0$$

MULTIPLICATION IS DENOTED BY

ab

AND IS REQUIRED TO BE DISTRIBUTIVE WITH RESPECT TO ADDITION

$$(a + b)c = ac + bc, \quad a(b + c) = ab + ac$$

IMPORTANT: A RING MAY OR MAY NOT HAVE AN IDENTITY ELEMENT

$$1x = x1 = x$$

AN ALGEBRA (or RING) IS SAID TO BE ASSOCIATIVE (RESP. COMMUTATIVE) IF THE **MULTIPLICATION** IS ASSOCIATIVE (RESP. COMMUTATIVE)

(RECALL THAT ADDITION IS ALWAYS COMMUTATIVE AND ASSOCIATIVE)

Table 2

ALGEBRAS (OR RINGS)

commutative algebras

$$ab = ba$$

associative algebras

$$a(bc) = (ab)c$$

Lie algebras

$$a^2 = 0$$

$$(ab)c + (bc)a + (ca)b = 0$$

Jordan algebras

$$ab = ba$$

$$a(a^2b) = a^2(ab)$$

Sophus Lie (1842–1899)

Marius Sophus Lie was a Norwegian mathematician. He largely created the theory of continuous symmetry, and applied it to the study of geometry and differential equations.

Pascual Jordan (1902–1980)

Pascual Jordan was a German theoretical and mathematical physicist who made significant contributions to quantum mechanics and quantum field theory.

THE DERIVATIVE

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

DIFFERENTIATION IS A LINEAR PROCESS

$$(f+g)' = f' + g'$$

$$(cf)' = cf'$$

THE SET OF DIFFERENTIABLE FUNCTIONS FORMS AN ALGEBRA $\mathcal{D}$

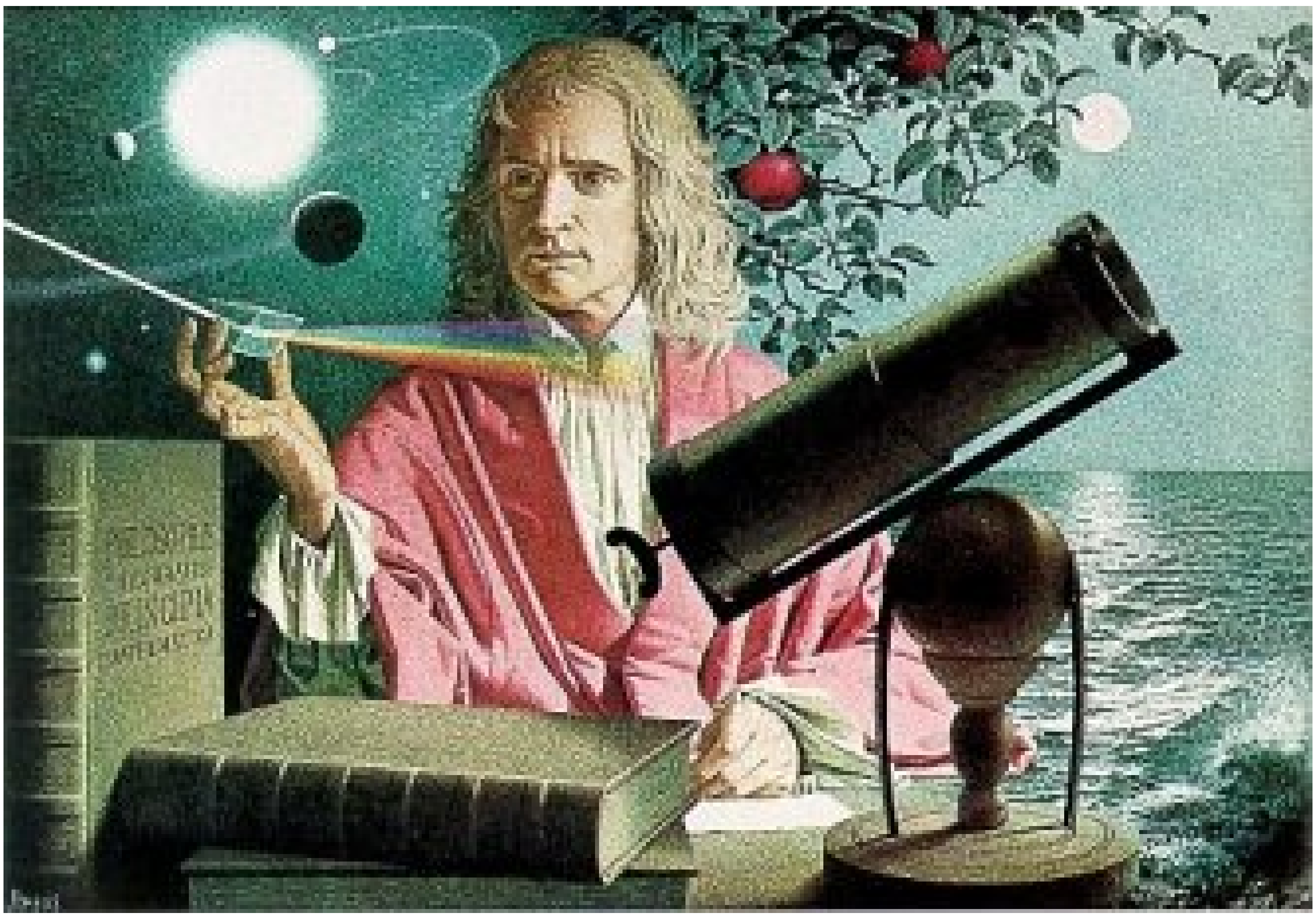
$$(fg)' = fg' + f'g$$

(product rule)

HEROS OF CALCULUS

#1

Sir Isaac Newton (1642-1727)



Isaac Newton was an English physicist, mathematician, astronomer, natural philosopher, alchemist, and theologian, and is considered by many scholars and members of the general public to be one of the most influential people in human history.

LEIBNIZ RULE

$$(fg)' = f'g + fg'$$

(order changed)

****

$$(fgh)' = f'gh + fg'h + fgh'$$

****

$$(f_1 f_2 \cdots f_n)' = f_1' f_2 \cdots f_n + \cdots + f_1 f_2 \cdots f_n'$$

The chain rule,

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

plays no role in this talk

Neither does the quotient rule

$$(f/g)' = \frac{gf' - fg'}{g^2}$$

CONTINUITY

$$x_n \to x \Rightarrow f(x_n) \to f(x)$$

THE SET OF CONTINUOUS FUNCTIONS FORMS AN ALGEBRA $\mathcal{C}$

(sums, constant multiples and products of continuous functions are continuous)

$\mathcal{D}$ and $\mathcal{C}$ ARE EXAMPLES OF ALGEBRAS WHICH ARE BOTH **ASSOCIATIVE** AND **COMMUTATIVE**

PROPOSITION 1
EVERY DIFFERENTIABLE FUNCTION IS CONTINUOUS

$\mathcal{D}$ is a subalgebra of $\mathcal{C}$; $\mathcal{D} \subset \mathcal{C}$

DIFFERENTIATION IS A LINEAR PROCESS

LET US DENOTE IT BY D AND WRITE Df for f'

$$D(f+g) = Df + Dg$$

$$D(cf) = cDf$$

$$D(fg) = (Df)g + f(Dg)$$

$$D(f/g) = \frac{g(Df) - f(Dg)}{g^2}$$

IS THE LINEAR PROCESS $D : f \mapsto f'$ CONTINUOUS?

(If $f_n \to f$ in $\mathcal{D}$, does it follow that $f'_n \to f'$?)

(ANSWER: NO!)

DEFINITION 1

A <u>DERIVATION</u> ON $\mathcal{C}$ IS A LINEAR PROCESS SATISFYING THE LEIBNIZ RULE:

$$\delta(f+g) = \delta(f) + \delta(g)$$

$$\delta(cf) = c\delta(f)$$

$$\delta(fg) = \delta(f)g + f\delta(g)$$

THEOREM 1

There are no (non-zero) derivations on $\mathcal{C}$.

In other words,
Every derivation of $\mathcal{C}$ is identically zero

COROLLARY $\mathcal{D} \neq \mathcal{C}$

(NO DUUUH! $f(x) = |x|$)

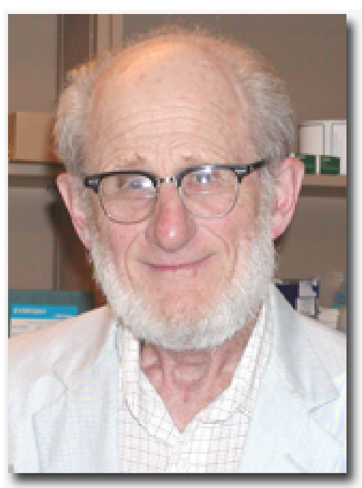
THEOREM 1A
(1955-Singer and Wermer)

Every continuous derivation on $\mathcal{C}$ is zero.

Theorem 1B
(1960-Sakai)

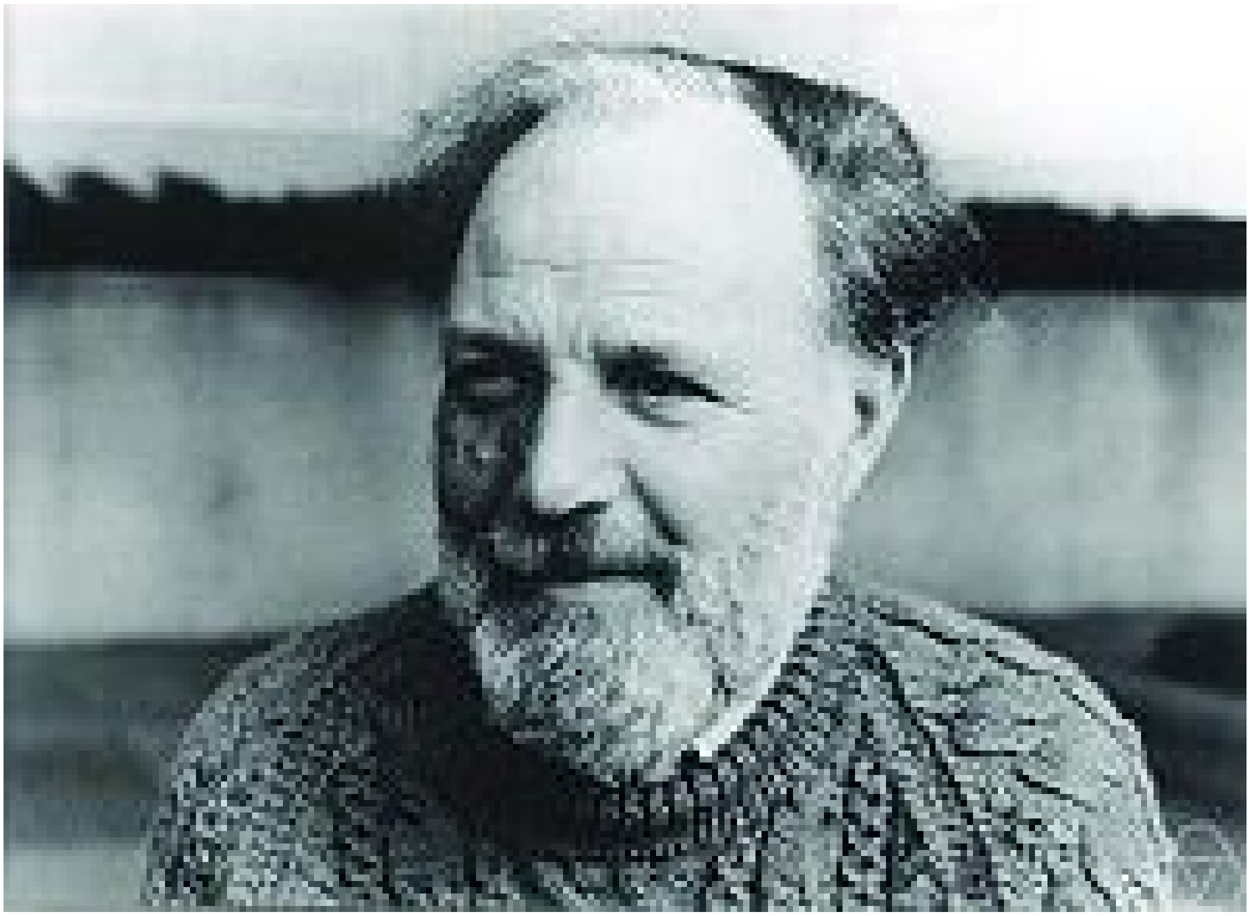
Every derivation on $\mathcal{C}$ is continuous.

(False for $\mathcal{D}$)



John Wermer (b. 1925) **Soichiro Sakai (b. 1926)**

Isadore Singer (b. 1924)



Isadore Manuel Singer is an Institute Professor in the Department of Mathematics at the Massachusetts Institute of Technology. He is noted for his work with Michael Atiyah in 1962, which paved the way for new interactions between pure mathematics and theoretical physics.

DERIVATIONS ON THE SET OF MATRICES

THE SET $M_n(\mathbf{R})$ of n by n MATRICES IS AN ALGEBRA UNDER

MATRIX ADDITION

$$A + B$$

AND

MATRIX MULTIPLICATION

$$A \times B$$

WHICH IS ASSOCIATIVE BUT NOT COMMUTATIVE.

(WE SHALL DEFINE TWO MORE MULTIPLICATIONS)

DEFINITION 2

A DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO MATRIX MULTIPLICATION IS A LINEAR PROCESS δ WHICH SATISFIES THE PRODUCT RULE

$$\delta(A \times B) = \delta(A) \times B + A \times \delta(B)$$

.

PROPOSITION 2

FIX A MATRIX A in $M_n(\mathbf{R})$ AND DEFINE

$$\delta_A(X) = A \times X - X \times A.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO MATRIX MULTIPLICATION (WHICH CAN BE NON-ZERO)

THEOREM 2

(1942 Hochschild)

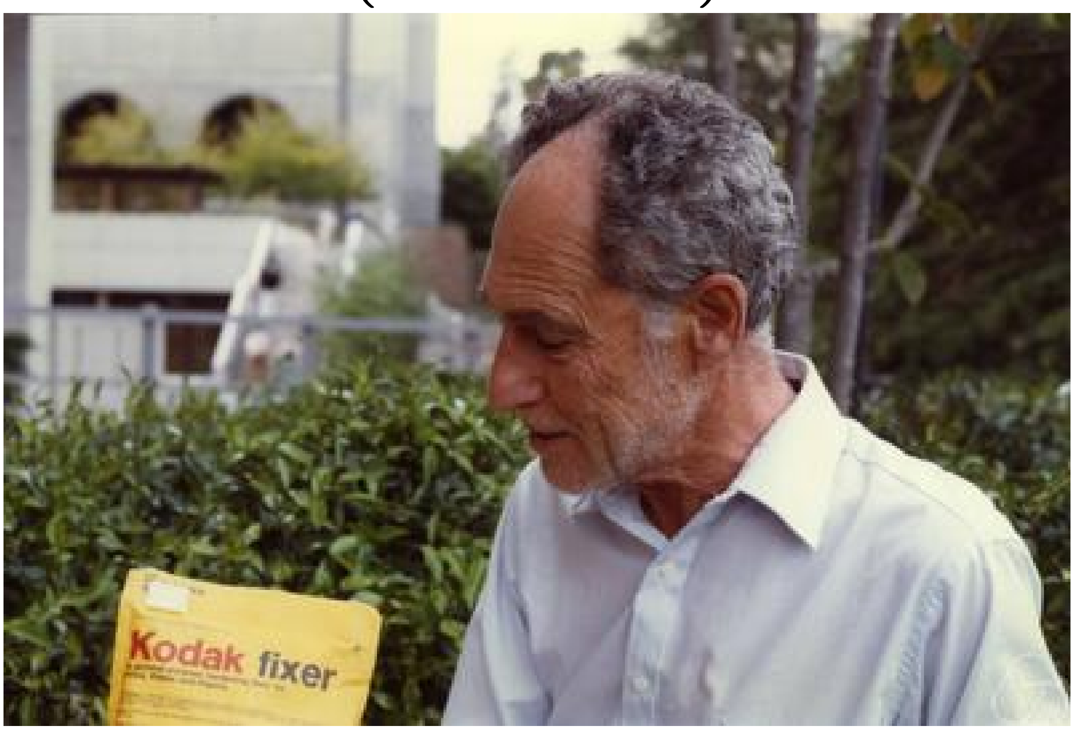
EVERY DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO MATRIX MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbf{R})$.

Gerhard Hochschild (1915–2010)

(Photo 1968)

Gerhard Paul Hochschild was an American mathematician who worked on Lie groups, algebraic groups, homological algebra and algebraic number theory.

(Photo 1976)



(Photo 1981)

Joseph Henry Maclagan Wedderburn (1882–1948)

Scottish mathematician, who taught at Princeton University for most of his career. A significant algebraist, he proved that a finite division algebra is a field, and part of the Artin–Wedderburn theorem on simple algebras. He also worked on group theory and matrix algebra.

Amalie Emmy Noether (1882–1935)

Amalie Emmy Noether was an influential German mathematician known for her groundbreaking contributions to abstract algebra and theoretical physics. Described as the most important woman in the history of mathematics, she revolutionized the theories of rings, fields, and algebras. In physics, Noether's theorem explains the fundamental connection between symmetry and conservation laws.

THE BRACKET PRODUCT ON THE SET OF MATRICES

(THIS IS THE SECOND MULTIPLICATION)

THE BRACKET PRODUCT ON THE SET $M_n(\mathbf{R})$ OF MATRICES IS DEFINED BY

$$[X, Y] = X \times Y - Y \times X$$

THE SET $M_n(\mathbf{R})$ of n by n MATRICES IS AN ALGEBRA UNDER MATRIX ADDITION AND BRACKET MULTIPLICATION, WHICH IS NOT ASSOCIATIVE AND NOT COMMUTATIVE.

DEFINITION 3

A DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO BRACKET MULTIPLICATION

IS A LINEAR PROCESS δ WHICH SATISFIES THE PRODUCT RULE

$$\delta([A, B]) = [\delta(A), B] + [A, \delta(B)]$$

.

PROPOSITION 3

FIX A MATRIX A in $M_n(\mathbf{R})$ AND DEFINE

$$\delta_A(X) = [A, X] = A \times X - X \times A.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO BRACKET MULTIPLICATION

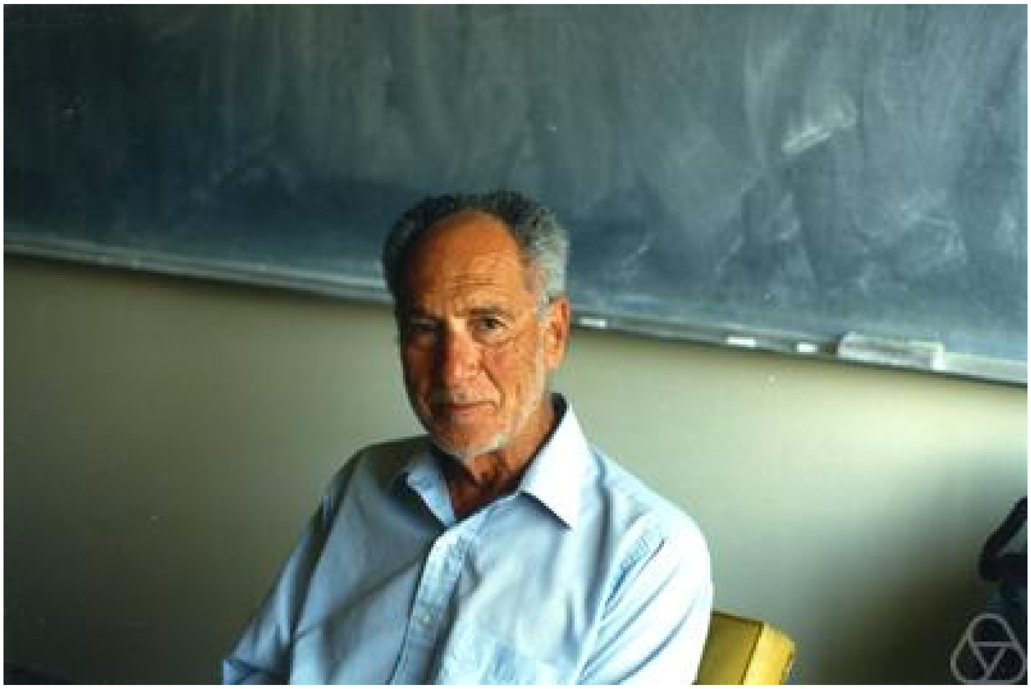
THEOREM 3

(1942 Hochschild, Zassenhaus)
EVERY DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO BRACKET MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbf{R})$.

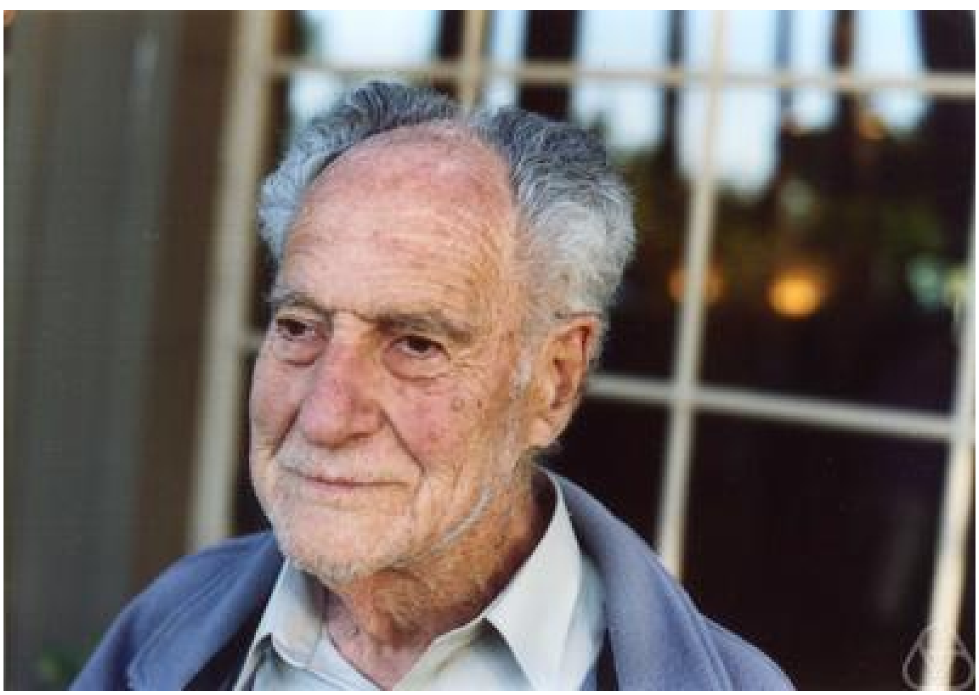
Hans Zassenhaus (1912–1991)

Hans Julius Zassenhaus was a German mathematician, known for work in many parts of abstract algebra, and as a pioneer of computer algebra.

Gerhard Hochschild (1915–2010)



(Photo 1986)



(Photo 2003)

THE CIRCLE PRODUCT ON THE SET OF MATRICES

(THIS IS THE THIRD MULTIPLICATION)

THE CIRCLE PRODUCT ON THE SET $M_n(\mathbf{R})$ OF MATRICES IS DEFINED BY

$$X \circ Y = (X \times Y + Y \times X)/2$$

THE SET $M_n(\mathbf{R})$ of n by n MATRICES IS AN ALGEBRA UNDER MATRIX ADDITION AND CIRCLE MULTIPLICATION, WHICH IS COMMUTATIVE BUT NOT ASSOCIATIVE.

DEFINITION 4

A <u>DERIVATION</u> ON $M_n(\mathbf{R})$ WITH <u>RESPECT TO CIRCLE MULTIPLICATION</u>

IS A LINEAR PROCESS δ WHICH SATISFIES THE PRODUCT RULE

$$\delta(A \circ B) = \delta(A) \circ B + A \circ \delta(B)$$

PROPOSITION 4

FIX A MATRIX A in $M_n(\mathbf{R})$ AND DEFINE

$$\delta_A(X) = A \times X - X \times A.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO CIRCLE MULTIPLICATION

THEOREM 4

(1972-Sinclair)

EVERY DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO CIRCLE MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbf{R})$.

REMARK

(1937-Jacobson)

THE ABOVE PROPOSITION AND THEOREM NEED TO BE MODIFIED FOR THE SUBALGEBRA (WITH RESPECT TO CIRCLE MULTIPLICATION) OF SYMMETRIC MATRICES.

Alan M. Sinclair (retired)

Nathan Jacobson (1910–1999)

Nathan Jacobson was an American mathematician who was recognized as one of the leading algebraists of his generation, and he was also famous for writing more than a dozen standard monographs.

Table 1

$M_n(\mathbf{R})$ (ALGEBRAS)

matrix	bracket	circle
$ab = a \times b$	$[a,b] = ab - ba$	$a \circ b = ab + ba$
Th. 2	Th.3	Th.4
$\delta_a(x)$ $=$ $ax - xa$	$\delta_a(x)$ $=$ $ax - xa$	$\delta_a(x)$ $=$ $ax - xa$

GRADUS AD PARNASSUM
PART I—ALGEBRAS

1. Prove Proposition 2
2. Prove Proposition 3
3. Prove Proposition 4
4. Let A, B are two fixed matrices in $M_n(\mathbf{R})$. Show that the linear process

$$\delta_{A,B}(X) = A \circ (B \circ X) - B \circ (A \circ X)$$

   is a derivation of $M_n(\mathbf{R})$ with respect to circle multiplication.
   (cf. Remark following Theorem 4)
5. Show that $M_n(\mathbf{R})$ is a Lie algebra with respect to bracket multiplication. In other words, show that the two axioms for Lie algebras in Table 2 are satisfied if ab denotes $[a, b] = ab - ba$ (a and b denote matrices and ab denotes matrix multiplication)

6. Show that $M_n(\mathbf{R})$ is a Jordan algebra with respect to circle multiplication. In other words, show that the two axioms for Jordan algebras in Table 2 are satisfied if ab denotes $a \circ b = ab + ba$ (a and b denote matrices and ab denotes matrix multiplication—forget about dividing by 2)
7. (Extra credit)
   Let us write $\delta_{a,b}$ for the linear process $\delta_{a,b}(x) = a(bx) - b(ax)$ in a Jordan algebra. Show that $\delta_{a,b}$ is a derivation of the Jordan algebra by following the outline below. (cf. Homework problem 4 above.)

   (a) In the Jordan algebra axiom

   $$u(u^2v) = u^2(uv),$$

   replace u by $u + w$ to obtain the two equations

   $$2u((uw)v) + w(u^2v) = 2(uw)(uv) + u^2(wv) \tag{1}$$

   and (**correcting the misprint in part I**)

   $$u(w^2v) + 2w((uw)v) = w^2(uv) + 2(uw)(\mathbf{w}v).$$

(Hint: Consider the “degree” of w on each side of the equation resulting from the substitution)

(b) In (1), interchange v and w and subtract the resulting equation from (1) to obtain the equation

$$2u(\delta_{v,w}(u)) = \delta_{v,w}(u^2). \qquad (2)$$

(c) In (2), replace u by $x + y$ to obtain the equation

$$\delta_{v,w}(xy) = y\delta_{v,w}(x) + x\delta_{v,w}(y),$$

which is the desired result.

END OF REVIEW OF PART I

2. REVIEW OF PART II

IN THESE TALKS, I AM MOSTLY INTERESTED IN NONASSOCIATIVE ALGEBRAS (PART I) AND NONASSOCIATIVE TRIPLE SYSTEMS (PART II), ALTHOUGH THEY MAY OR MAY NOT BE COMMUTATIVE.

(ASSOCIATIVE AND COMMUTATIVE HAVE TO BE INTERPRETED APPROPRIATELY FOR THE TRIPLE SYSTEMS CONSIDERED WHICH ARE NOT ACTUALLY ALGEBRAS)

DERIVATIONS ON RECTANGULAR MATRICES

MULTIPLICATION DOES NOT MAKE SENSE ON $M_{m,n}(\mathbf{R})$ if $m \neq n$.

NOT TO WORRY!

WE CAN FORM A TRIPLE PRODUCT

$$X \times Y^t \times Z$$

(TRIPLE MATRIX MULTIPLICATION)

COMMUTATIVE AND ASSOCIATIVE DON'T MAKE SENSE HERE. RIGHT?

WRONG!!

$$(X \times Y^t \times Z) \times A^t \times B = X \times Y^t \times (Z \times A^t \times B)$$

(WHAT WOULD ASSOCIATIVE MEAN FOR A "QUADRUPLE" PRODUCT?)

DEFINITION 5

A <u>DERIVATION</u> ON $M_{m,n}(\mathbf{R})$ WITH <u>RESPECT TO</u>
<u>TRIPLE MATRIX MULTIPLICATION</u>

IS A LINEAR PROCESS δ WHICH SATISFIES THE (TRIPLE) PRODUCT RULE

$$\delta(A \times B^t \times C) =$$
$$\delta(A) \times B^t \times C + A \times \delta(B)^t \times C + A \times B^t \times \delta(C)$$

PROPOSITION 5

FOR TWO MATRICES A, B in $M_{m,n}(\mathbf{R})$,

DEFINE $\delta_{A,B}(X) =$

$$A \times B^t \times X + X \times B^t \times A - B \times A^t \times X - X \times A^t \times B$$

THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE MATRIX MULTIPLICATION

THEOREM 8*

EVERY DERIVATION ON $M_{m,n}(\mathbf{R})$ WITH RESPECT TO TRIPLE MATRIX MULTIPLICATION IS A **SUM** OF DERIVATIONS OF THE FORM $\delta_{A,B}$.

REMARK

THESE RESULTS HOLD TRUE AND ARE OF INTEREST FOR THE CASE $m = n$.

(WE SHALL DEFINE TWO OTHER TRIPLE PRODUCTS)

*Theorems 5,6,7 were in part I

TRIPLE BRACKET MULTIPLICATION

LET'S GO BACK FOR A MOMENT TO SQUARE MATRICES AND THE BRACKET MULTIPLICATION.

MOTIVATED BY THE LAST REMARK, WE DEFINE THE TRIPLE BRACKET MULTIPLICATION TO BE $[[X, Y], Z]$

(THIS IS THE SECOND TRIPLE PRODUCT)

DEFINITION 6

A <u>DERIVATION</u> ON $M_n(\mathbf{R})$ WITH <u>RESPECT TO</u> <u>TRIPLE BRACKET MULTIPLICATION</u>

IS A LINEAR PROCESS δ WHICH SATISFIES THE TRIPLE PRODUCT RULE

$$\delta([[A, B], C]) = [[\delta(A), B], C] + [[A, \delta(B)], C] + [[A, B], \delta(C)]$$

PROPOSITION 6

FIX TWO MATRICES A, B IN $M_n(\mathbf{R})$ AND DEFINE $\delta_{A,B}(X) = [[A, B], X]$

THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE BRACKET MULTIPLICATION.

THEOREM 9

EVERY DERIVATION OF $M_n(\mathbf{R})$ WITH RESPECT TO TRIPLE BRACKET MULTIPLICATION IS A SUM OF DERIVATIONS OF THE FORM $\delta_{A,B}$.

TRIPLE CIRCLE MULTIPLICATION

LET'S RETURN TO RECTANGULAR MATRICES AND FORM THE TRIPLE CIRCLE MULTIPLICATION

$$(A \times B^t \times C + C \times B^t \times A)/2$$

For sanity's sake, let us write this as

$$\{A, B, C\} = (A \times B^t \times C + C \times B^t \times A)/2$$

(THIS IS THE THIRD TRIPLE PRODUCT)

DEFINITION 7

A DERIVATION ON $M_{m,n}(\mathbf{R})$ WITH RESPECT TO TRIPLE CIRCLE MULTIPLICATION

IS A LINEAR PROCESS δ WHICH SATISFIES THE TRIPLE PRODUCT RULE

$$\delta(\{A,B,C\}) = \{\delta(A), B, C\} + \{A, \delta(B), C\} + \{A, B, \delta(C)\}$$

PROPOSITION 7

FIX TWO MATRICES A, B IN $M_{m,n}(\mathbf{R})$ AND DEFINE

$$\delta_{A,B}(X) = \{A, B, X\} - \{B, A, X\}$$

THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE CIRCLE MULTIPLICATION.

THEOREM 10

EVERY DERIVATION OF $M_{m,n}(\mathbf{R})$ WITH RESPECT TO TRIPLE CIRCLE MULTIPLICATION IS A **SUM** OF DERIVATIONS OF THE FORM $\delta_{A,B}$.

IT IS TIME FOR ANOTHER SUMMARY OF THE PRECEDING

Table 3

$M_{m,n}(\mathbf{R})$ (TRIPLE SYSTEMS)

triple matrix	triple bracket	triple circle
ab^tc	$[[a,b],c]$	$ab^tc + cb^ta$
Th. 8	Th.9	Th.10
$\delta_{a,b}(x)$ $=$ ab^tx $+xb^ta$ $-ba^tx$ $-xa^tb$	$\delta_{a,b}(x)$ $=$ abx $+xba$ $-bax$ $-xab$	$\delta_{a,b}(x)$ $=$ ab^tx $+xb^ta$ $-ba^tx$ $-xa^tb$
(sums)	(sums) $(m = n)$	(sums)

(WHAT IS THE DEFINITION OF A DERIVATION OF A "QUADRUPLE" PRODUCT?)

LET'S PUT ALL THIS NONSENSE TOGETHER

Table 1 $M_n(\mathbf{R})$ (ALGEBRAS)

matrix	bracket	circle
$ab = a \times b$	$[a, b] = ab - ba$	$a \circ b = ab + ba$
Th. 2	Th.3	Th.4
$\delta_a(x) = ax - xa$	$\delta_a(x) = ax - xa$	$\delta_a(x) = ax - xa$

Table 3 $M_{m,n}(\mathbf{R})$ (TRIPLE SYSTEMS)

triple matrix	triple bracket	triple circle
ab^tc	$[[a, b], c]$	$ab^tc + cb^ta$
Th. 8	Th.9	Th.10
$\delta_{a,b}(x) = ab^tx + xb^ta - ba^tx - xa^tb$	$\delta_{a,b}(x) = abx + xba - bax - xab$	$\delta_{a,b}(x) = ab^tx + xb^ta - ba^tx - xa^tb$
(sums)	(sums) $(m = n)$	(sums)

HEY! IT IS NOT SO NONSENSICAL!

AXIOMATIC APPROACH FOR TRIPLE SYSTEMS

AN TRIPLE SYSTEM IS DEFINED TO BE A SET (ACTUALLY A VECTOR SPACE) WITH ONE BINARY OPERATION, CALLED ADDITION AND ONE TERNARY OPERATION CALLED TRIPLE MULTIPLICATION

ACTUALLY, IF YOU FORGET ABOUT THE VECTOR SPACE, THIS DEFINES A

TERNARY RING

ADDITION IS DENOTED BY

$$a + b$$

AND IS REQUIRED TO BE COMMUTATIVE AND ASSOCIATIVE

$$a + b = b + a, \quad (a + b) + c = a + (b + c)$$

(THIS IS EXACTLY THE SAME AS FOR ALGEBRAS, OR RINGS, INCLUDING THE EXISTENCE OF 0)

TRIPLE MULTIPLICATION IS DENOTED

$$abc$$

AND IS REQUIRED TO BE LINEAR IN EACH VARIABLE

$$(a + b)cd = acd + bcd$$
$$a(b + c)d = abd + acd$$
$$ab(c + d) = abc + abd$$

AXIOMATIC APPROACH FOR TRIPLE SYSTEMS

THE AXIOM WHICH CHARACTERIZES TRIPLE MATRIX MULTIPLICATION IS

$$(abc)de = ab(cde) = a(dcb)e$$

THESE ARE CALLED

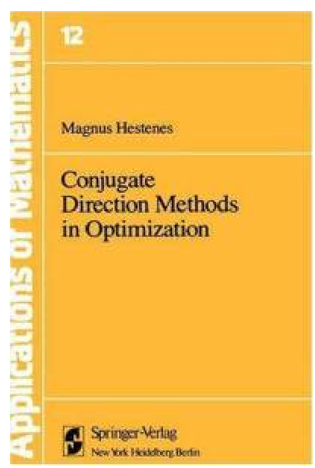
ASSOCIATIVE TRIPLE SYSTEMS

or

HESTENES ALGEBRAS

Magnus Hestenes (1906–1991)

Magnus Rudolph Hestenes was an American mathematician. Together with Cornelius Lanczos and Eduard Stiefel, he invented the conjugate gradient method.



THE AXIOMS WHICH CHARACTERIZE TRIPLE BRACKET MULTIPLICATION ARE

$$aab = 0$$

$$abc + bca + cab = 0$$

$$de(abc) = (dea)bc + a(deb)c + ab(dec)$$

THESE ARE CALLED
LIE TRIPLE SYSTEMS

(NATHAN JACOBSON, MAX KOECHER)

Max Koecher (1924–1990)

Max Koecher was a German mathematician. His main research area was the theory of Jordan algebras, where he introduced the KantorKoecherTits construction.

Nathan Jacobson (1910–1999)

THE AXIOMS WHICH CHARACTERIZE TRIPLE CIRCLE MULTIPLICATION ARE

$$abc = cba$$

$$de(abc) = (dea)bc - a(edb)c + ab(dec)$$

THESE ARE CALLED

JORDAN TRIPLE SYSTEMS

Kurt Meyberg (living)

Ottmar Loos + Erhard Neher (both living)

YET ANOTHER SUMMARY

Table 4

TRIPLE SYSTEMS

associative triple systems

$$(abc)de = ab(cde) = a(dcb)e$$

Lie triple systems

$$aab = 0$$

$$abc + bca + cab = 0$$

$$de(abc) = (dea)bc + a(deb)c + ab(dec)$$

Jordan triple systems

$$abc = cba$$

$$de(abc) = (dea)bc - a(edb)c + ab(dec)$$

GRADUS AD PARNASSUM
PART II—TRIPLE SYSTEMS

1. Prove Proposition 5
   (Use the notation $\langle abc\rangle$ for ab^tc)
2. Prove Proposition 6
   (Use the notation $[abc]$ for $[[a,b],c]$)
3. Prove Proposition 7
   (Use the notation $\{abc\}$ for $ab^tc + cb^ta$)
4. Show that $M_n(\mathbf{R})$ is a Lie triple system with respect to triple bracket multiplication. In other words, show that the three axioms for Lie triple systems in Table 4 are satisfied if abc denotes $[[a,b],c] = (ab - ba)c - c(ab - ba)$ (a, b and c denote matrices)
   (Use the notation $[abc]$ for $[[a,b],c]$)
5. Show that $M_{m,n}(\mathbf{R})$ is a Jordan triple system with respect to triple circle multiplication. In other words, show that the two axioms for Jordan triple systems in Table 4 are satisfied if abc denotes $ab^tc + cb^ta$ (a, b and c denote matrices)
   (Use the notation $\{abc\}$ for $ab^tc + cb^ta$)

6. Let us write $\delta_{a,b}$ for the linear process

$$\delta_{a,b}(x) = abx$$

in a Lie triple system. Show that $\delta_{a,b}$ is a derivation of the Lie triple system by using the axioms for Lie triple systems in Table 4.

(Use the notation $[abc]$ for the triple product in any Lie triple system, so that, for example, $\delta_{a,b}(x)$ is denoted by $[abx]$)

7. Let us write $\delta_{a,b}$ for the linear process

$$\delta_{a,b}(x) = abx - bax$$

in a Jordan triple system. Show that $\delta_{a,b}$ is a derivation of the Jordan triple system by using the axioms for Jordan triple systems in Table 4.

(Use the notation $\{abc\}$ for the triple product in any Jordan triple system, so that, for example, $\delta_{a,b}(x) = \{abx\} - \{bax\}$)

8. On the Jordan algebra $M_n(\mathbf{R})$ with the circle product $a \circ b = ab + ba$, define a triple product

$$\{abc\} = (a \circ b) \circ c + (c \circ b) \circ a - (a \circ c) \circ b.$$

Show that $M_n(\mathbf{R})$ is a Jordan triple system with this triple product.
Hint: show that $\{abc\} = 2a \times b \times c + 2c \times b \times a$

9. On the vector space $M_n(\mathbf{R})$, define a triple product $\langle abc \rangle = abc$ (matrix multiplication without the transpose in the middle). Formulate the definition of a derivation of the resulting triple system, and state and prove a result corresponding to Proposition 5. Is this triple system associative?

10. In an associative algebra, define a triple product $\langle abc \rangle$ to be $(ab)c$. Show that the algebra becomes an associative triple system with this triple product.

11. In an associative triple system with triple product denoted $\langle abc \rangle$, define a binary product ab to be $\langle aub \rangle$, where u is a fixed element. Show that the triple system becomes an associative algebra with this product.

12. In a Lie algebra with product denoted by $[a, b]$, define a triple product $[abc]$ to be $[[a, b], c]$. Show that the Lie algebra becomes a Lie triple system with this triple product.
13. Let A be an algebra (associative, Lie, or Jordan; it doesn't matter). Show that the set $\mathcal{D} := \text{Der}\,(A)$ of all derivations of A is a Lie subalgebra of $\text{End}\,(A)$. That is $\mathcal{D}$ is a linear subspace of the vector space of linear transformations on A, and if $D_1, D_2 \in \mathcal{D}$, then $D_1D_2 - D_2D_1 \in \mathcal{D}$.
14. Let A be a triple system (associative, Lie, or Jordan; it doesn't matter). Show that the set $\mathcal{D} := \text{Der}\,(A)$ of derivations of A is a Lie subalgebra of $\text{End}\,(A)$. That is $\mathcal{D}$ is a linear subspace of the vector space of linear transformations on A, and if $D_1, D_2 \in \mathcal{D}$, then $D_1D_2 - D_2D_1 \in \mathcal{D}$.

END OF REVIEW OF PART II

GRADUS AD PARNASSUM
PART III
ALGEBRAS AND TRIPLE SYSTEMS
(SNEAK PREVIEW)

1. In an arbitrary Jordan triple system, with triple product denoted by $\{abc\}$, define a triple product by

$$[abc] = \{abc\} - \{bac\}.$$

   Show that the Jordan triple system becomes a Lie triple system with this new triple product.

2. In an arbitrary associative triple system, with triple product denoted by $\langle abc\rangle$, define a triple product by

$$[xyz] = \langle xyz\rangle - \langle yxz\rangle - \langle zxy\rangle + \langle zyx\rangle.$$

   Show that the associative triple system becomes a Lie triple system with this new triple product.

3. In an arbitrary Jordan algebra, with product denoted by xy, define a triple product by $[xyz] = x(yz) - y(xz)$. Show that the Jordan algebra becomes a Lie triple system with this new triple product.
4. In an arbitrary Jordan triple system, with triple product denoted by $\{abc\}$, fix an element y and define a binary product by

$$ab = \{ayb\}.$$

Show that the Jordan triple system becomes a Jordan algebra with this (binary) product.
5. In an arbitrary Jordan algebra with multiplication denoted by ab, define a triple product

$$\{abc\} = (ab)c + (cb)a - (ac)b.$$

Show that the Jordan algebra becomes a Jordan triple system with this triple product. (cf. Problem 8)

6. Show that every Lie triple system, with triple product denoted $[abc]$ is a subspace of some Lie algebra, with product denoted $[a, b]$, such that $[abc] = [[a, b], c]$.
7. Find out what a semisimple associative algebra is and prove that every derivation of a finite dimensional semisimple associative algebra is inner, that is, of the form $x \mapsto ax - xa$ for some fixed a in the algebra.
8. Find out what a semisimple Lie algebra is and prove that every derivation of a finite dimensional semisimple Lie algebra is inner, that is, of the form $x \mapsto [a, x]$ for some fixed a in the algebra.
9. Find out what a semisimple Jordan algebra is and prove that every derivation of a finite dimensional semisimple Jordan algebra is inner, that is, of the form $x \mapsto \sum_{i=1}^{n}(a_i(b_i x) - b_i(a_i x))$ for some fixed elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in the algebra.

10. Find out what a semisimple associative triple system is and prove that every derivation of a finite dimensional semisimple associative triple system is inner, that is, of the form $x \mapsto \sum_{i=1}^{n}(\langle a_i b_i x\rangle - \langle b_i a_i x\rangle)$ for some fixed elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in the associative triple system.
11. Find out what a semisimple Lie triple system is and prove that every derivation of a finite dimensional semisimple Lie triple system is inner, that is, of the form $x \mapsto \sum_{i=1}^{n}[a_i b_i x]$ for some fixed elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in the Lie triple system.
12. Find out what a semisimple Jordan triple system is and prove that every derivation of a finite dimensional semisimple Jordan triple system is inner, that is, of the form $x \mapsto \sum_{i=1}^{n}(\{a_i b_i x\} - \{b_i a_i x\})$ for some fixed elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in the Jordan triple system.

3. WHAT IS A MODULE?

The American Heritage Dictionary of the English Language, Fourth Edition 2009.

1. A standard or unit of measurement.
2. **Architecture** The dimensions of a structural component, such as the base of a column, used as a unit of measurement or standard for determining the proportions of the rest of the construction.
3. **Visual Arts/Furniture** A standardized, often interchangeable component of a system or construction that is designed for easy assembly or flexible use: a sofa consisting of two end modules.
4. **Electronics** A self-contained assembly of electronic components and circuitry, such as a stage in a computer, that is installed as a unit.

5. **Computer Science** A portion of a program that carries out a specific function and may be used alone or combined with other modules of the same program.
6. **Astronautics** A self-contained unit of a spacecraft that performs a specific task or class of tasks in support of the major function of the craft.
7. **Education** A unit of education or instruction with a relatively low student-to-teacher ratio, in which a single topic or a small section of a broad topic is studied for a given period of time.
8. **Mathematics** A system with scalars coming from a ring.

Nine Zulu Queens Ruled China

- Mathematicians think of numbers as a set of nested Russian dolls. The inhabitants of each Russian doll are honorary inhabitants of the next one out.

$$\mathbf{N} \subset \mathbf{Z} \subset \mathbf{Q} \subset \mathbf{R} \subset \mathbf{C}$$

- In **N** you can't subtract; in **Z** you can't divide; in **Q** you can't take limits; in **R** you can't take the square root of a negative number. With the complex numbers **C**, nothing is impossible. You can even raise a number to a complex power.

- $\mathbf{Z}$ is a ring
- $\mathbf{Q}, \mathbf{R}, \mathbf{C}$ are fields
- $\mathbf{Q}^n$ is a vector space over $\mathbf{Q}$
- $\mathbf{R}^n$ is a vector space over $\mathbf{R}$
- $\mathbf{C}^n$ is a vector space over $\mathbf{C}$

A **field** is a commutative ring with identity element 1 such that for every nonzero element x, there is an element called x^{-1} such that

$$xx^{-1} = 1$$

A **vector space** over a field F (called the field of scalars) is a set V with an addition $+$ which is commutative and associative and has a zero element and for which there is a "scalar" product ax in V for each a in F and x in V, satisfying the following properties for arbitrary elements a, b in F and x, y in V:

1. $(a + b)x = ax + bx$
2. $a(x + y) = ax + ay$
3. $a(bx) = (ab)x$
4. $1x = x$

In abstract algebra, the concept of a module over a ring is a generalization of the notion of **vector space**, wherein the corresponding scalars are allowed to lie in an arbitrary ring.

Modules also generalize the notion of **abelian groups**, which are modules over the ring of integers.

Thus, a module, like a vector space, is an additive abelian group; a product is defined between elements of the ring and elements of the module, and this multiplication is associative (when used with the multiplication in the ring) and distributive.

Modules are very closely related to the
representation theory
of groups and of other algebraic structures.

They are also one of the central notions of

commutative algebra
and
homological algebra,

and are used widely in

algebraic geometry
and
algebraic topology.

The traditional division of mathematics into subdisciplines:

Arithmetic (whole numbers)
Geometry (figures)
Algebra (abstract symbols)
Analysis (limits).

MATHEMATICS SUBJECT CLASSIFICATION

(AMERICAN MATHEMATICAL SOCIETY)

00-XX General
01-XX History and biography
03-XX Mathematical logic and foundations
05-XX Combinatorics
06-XX Lattices, ordered algebraic structures
08-XX General algebraic systems
11-XX Number Theory
12-XX Field theory and polynomials
13-XX **COMMUTATIVE ALGEBRA**
14-XX **ALGEBRAIC GEOMETRY**
15-XX Linear algebra; matrix theory
16-XX Associative rings and algebras
16-XX **REPRESENTATION THEORY**
17-XX Nonassociative rings and algebras
18-XX Category theory;
18-XX **HOMOLOGICAL ALGEBRA**
19-XX K-theory
20-XX Group theory and generalizations
20-XX **REPRESENTATION THEORY**
22-XX Topological groups, Lie groups

26-XX Real functions
28-XX Measure and integration
30-XX Complex Function Theory
31-XX Potential theory
32-XX Several complex variables
33-XX Special functions
34-XX Ordinary differential equations
35-XX Partial differential equations
37-XX Dynamical systems, ergodic theory
39-XX Difference and functional equations
40-XX Sequences, series, summability
41-XX Approximations and expansions
42-XX Harmonic analysis on Euclidean spaces
43-XX Abstract harmonic analysis
44-XX Integral transforms
45-XX Integral equations
46-XX Functional analysis
47-XX Operator theory
49-XX Calculus of variations, optimal control
51-XX Geometry
52-XX Convex and discrete geometry
53-XX Differential geometry
54-XX General topology

55-XX **ALGEBRAIC TOPOLOGY**
57-XX Manifolds and cell complexes
58-XX Global analysis, analysis on manifolds
60-XX Probability theory
62-XX Statistics
65-XX Numerical analysis
68-XX Computer science
70-XX Mechanics of particles and systems
74-XX Mechanics of deformable solids
76-XX Fluid mechanics
78-XX Optics, electromagnetic theory
80-XX Classical thermodynamics, heat
81-XX Quantum theory
82-XX Statistical mechanics, matter
83-XX Relativity and gravitational theory
85-XX Astronomy and astrophysics
86-XX Geophysics
90-XX Operations research
91-XX Game theory, economics
92-XX Biology and other natural sciences
93-XX Systems theory; control
94-XX Information and communication
97-XX Mathematics education

MOTIVATION

In a vector space, the set of scalars forms a field and acts on the vectors by scalar multiplication, subject to certain axioms such as the distributive law. In a module, the scalars need only be a ring, so the module concept represents a significant generalization.

In commutative algebra, it is important that both ideals and quotient rings are modules, so that many arguments about ideals or quotient rings can be combined into a single argument about modules.

In non-commutative algebra the distinction between left ideals, ideals, and modules becomes more pronounced, though some important ring theoretic conditions can be expressed either about left ideals or left modules.

Much of the theory of modules consists of extending as many as possible of the desirable properties of vector spaces to the realm of modules over a ”well-behaved” ring, such as a principal ideal domain.

However, modules can be quite a bit more complicated than vector spaces; for instance, not all modules have a basis, and even those that do, **free modules**, need not have a unique rank if the underlying ring does not satisfy the invariant basis number condition.

Vector spaces always have a basis whose cardinality is unique (assuming the axiom of choice).

FORMAL DEFINITION

A left R-module M over the ring R consists of an abelian group (M, $+$) and an operation $R \times M \to M$ such that for all r,s in R, x,y in M, we have:

$$r(x + y) = rx + ry$$

$$(r + s)x = rx + sx$$

$$(rs)x = r(sx)$$

$$1x = x$$

if R has multiplicative identity 1.

The operation of the ring on M is called scalar multiplication, and is usually written by juxtaposition, i.e. as rx for r in R and x in M.

If one writes the scalar action as f_r so that $f_r(x) = rx$, and f for the map which takes each r to its corresponding map f_r, then the first axiom states that every f_r is a group homomorphism of M, and the other three axioms assert that the map f:R $\to$ End(M) given by $r \mapsto f_r$ is a ring homomorphism from R to the endomorphism ring End(M).

In this sense, module theory generalizes representation theory, which deals with group actions on vector spaces.

A **bimodule** is a module which is a left module and a right module such that the two multiplications are compatible.

EXAMPLES

1. If K is a field, then the concepts "K-vector space" (a vector space over K) and K-module are identical.
2. The concept of a Z-module agrees with the notion of an abelian group. That is, every abelian group is a module over the ring of integers Z in a unique way. For n $\geq$ 0, let nx = x + x + ... + x (n summands), 0x = 0, and (-n)x = -(nx). Such a module need not have a basis
3. If R is any ring and n a natural number, then the cartesian product R^n is both a left and a right module over R if we use the component-wise operations. Hence when n = 1, R is an R-module, where the scalar multiplication is just ring multiplication. The case n = 0 yields the trivial R-module 0 consisting only of its identity element. Modules of this type are called free

4. If S is a nonempty set, M is a left R-module, and M^S is the collection of all functions f : S $\rightarrow$ M, then with addition and scalar multiplication in M^S defined by (f + g)(s) = f(s) + g(s) and (rf)(s) = rf(s), M^S is a left R-module. The right R-module case is analogous. In particular, if R is commutative then the collection of R-module homomorphisms h : M $\rightarrow$ N (see below) is an R-module (and in fact a submodule of N^M).
5. The square n-by-n matrices with real entries form a ring R, and the Euclidean space R^n is a left module over this ring if we define the module operation via matrix multiplication. If R is any ring and I is any left ideal in R, then I is a left module over R. Analogously of course, right ideals are right modules.
6. There are modules of a Lie algebra as well.

SUBMODULES AND HOMOMORPHISMS

Suppose M is a left R-module and N is a subgroup of M. Then N is a **submodule** (or R-submodule, to be more explicit) if, for any n in N and any r in R, the product r n is in N (or nr for a right module).

If M and N are left R-modules, then a map $f : M \to N$ is a **homomorphism of R-modules** if, for any m, n in M and r, s in R, $f(rm + sn) = rf(m) + sf(n)$.

This, like any homomorphism of mathematical objects, is just a mapping which preserves the structure of the objects. Another name for a homomorphism of modules over R is an R-linear map.

A bijective module homomorphism is an **isomorphism of modules**, and the two modules are called isomorphic.

Two isomorphic modules are identical for all practical purposes, differing solely in the notation for their elements.

The kernel of a module homomorphism f : M $\rightarrow$ N is the submodule of M consisting of all elements that are sent to zero by f.

The isomorphism theorems familiar from groups and vector spaces are also valid for R-modules.

TYPES OF MODULES

(a) **Finitely generated** A module M is finitely generated if there exist finitely many elements $x_1, \ldots x_n$ in M such that every element of M is a linear combination of those elements with coefficients from the scalar ring R.

(b) **Cyclic module** A module is called a cyclic module if it is generated by one element.

(c) **Free** A free module is a module that has a basis, or equivalently, one that is isomorphic to a direct sum of copies of the scalar ring R. These are the modules that behave very much like vector spaces.

(d) **Projective** Projective modules are direct summands of free modules and share many of their desirable properties.

(e) **Injective** Injective modules are defined dually to projective modules.

(f) **Flat** A module is called flat if taking the tensor product of it with any short exact sequence of R modules preserves exactness.

(g) **Simple** A simple module S is a module that is not 0 and whose only submodules are 0 and S. Simple modules are sometimes called irreducible.

(h) **Semisimple** A semisimple module is a direct sum (finite or not) of simple modules. Historically these modules are also called completely reducible.

(i) **Indecomposable** An indecomposable module is a non-zero module that cannot be written as a direct sum of two non-zero submodules. Every simple module is indecomposable, but there are indecomposable modules which are not simple (e.g. uniform modules).

(j) **Faithful** A faithful module M is one where the action of each r $\neq$ 0 in R on M is nontrivial (i.e. rx $\neq$ 0 for some x in M). Equivalently, the annihilator of M is the zero ideal.

(k) **Noetherian**. A Noetherian module is a module which satisfies the ascending chain condition on submodules, that is, every increasing chain of submodules becomes stationary after finitely many steps. Equivalently, every submodule is finitely generated.

(l) **Artinian** An Artinian module is a module which satisfies the descending chain condition on submodules, that is, every decreasing chain of submodules becomes stationary after finitely many steps.

(m) **Graded** A graded module is a module decomposable as a direct sum M $= \oplus_x M_x$ over a graded ring R $= \oplus_x R_x$ such that $R_x M_y \subset M_{x+y}$ for all x and y.

(n) **Uniform** A uniform module is a module in which all pairs of nonzero submodules have nonzero intersection.

RELATION TO REPRESENTATION THEORY

If M is a left R-module, then the action of an element r in R is defined to be the map M $\rightarrow$ M that sends each x to rx (or xr in the case of a right module), and is necessarily a group endomorphism of the abelian group (M,+).

The set of all group endomorphisms of M is denoted $End_Z(M)$ and forms a ring under addition and composition, and sending a ring element r of R to its action actually defines a ring homomorphism from R to $End_Z(M)$.

Such a ring homomorphism R $\rightarrow End_Z(M)$ is called a representation of R over the abelian group M; an alternative and equivalent way of defining left R-modules is to say that a left R-module is an abelian group M together with a representation of R over it.

A representation is called faithful if and only if the map R $\rightarrow End_Z(M)$ is injective. In terms of modules, this means that if r is an element of R such that rx=0 for all x in M, then r=0.

END OF "MODULE" ON MODULES

4. DERIVATIONS INTO A MODULE

CONTEXTS

(i) ASSOCIATIVE ALGEBRAS
(ii) JORDAN ALGEBRAS
(iii) JORDAN TRIPLE SYSTEMS

Could also consider:

(ii’) LIE ALGEBRAS
(iii’) LIE TRIPLE SYSTEMS
(i’) ASSOCIATIVE TRIPLE SYSTEMS

(i) ASSOCIATIVE ALGEBRAS

derivation: $D(ab) = a \cdot Db + Da \cdot b$

inner derivation: ad x(a) $= x \cdot a - a \cdot x$
$(x \in M)$

THEOREM (Noether,Wedderburn) (early 20th century))
EVERY DERIVATION OF SEMISIMPLE ASSOCIATIVE ALGEBRA IS INNER, THAT IS, OF THE FORM $x \mapsto ax - xa$ FOR SOME a IN THE ALGEBRA

THEOREM (Hochschild 1942)
EVERY DERIVATION OF SEMISIMPLE ASSOCIATIVE ALGEBRA INTO A MODULE IS INNER, THAT IS, OF THE FORM $x \mapsto ax - xa$ FOR SOME a IN THE MODULE

THEOREM (1983-Haagerup)
EVERY C^*-ALGEBRA IS WEAKLY AMENABLE.

Uffe Haagerup b. 1950

Haagerup's research is in operator theory, and covers many subareas in the subject which are currently very active - random matrices, free probability, C*-algebras and applications to mathematical physics.

(ii) JORDAN ALGEBRAS

derivation: $D(a \circ b) = a \circ Db + Da \circ b$

inner derivation:

$\sum_i [L(x_i)L(a_i) - L(a_i)L(x_i)]$

$(x_i \in M, a_i \in A)$

$b \mapsto \sum_i [x_i \circ (a_i \circ b) - a_i \circ (x_i \circ b)]$

THEOREM (1949-Jacobson)
EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE JORDAN ALGEBRA INTO ITSELF IS INNER

THEOREM (1951-Jacobson)
EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE JORDAN ALGEBRA INTO A (JORDAN) **MODULE** IS INNER
(Lie algebras, Lie triple systems)

(iii) JORDAN TRIPLE SYSTEMS

derivation:

$$D\{a,b,c\} = \{Da.b,c\} + \{a,Db,c\} + \{a,b,Dc\}$$

$$\{x,y,z\} = (xy^*z + zy^*x)/2$$

inner derivation: $\sum_i [L(x_i,a_i) - L(a_i,x_i)]$

$$(x_i \in M, a_i \in A)$$

$$b \mapsto \sum_i [\{x_i,a_i,b\} - \{a_i,x_i,b\}]$$

THEOREM (1972 Meyberg)

EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE JORDAN TRIPLE SYSTEM IS INNER

(Lie algebras, Lie triple systems)

THEOREM (1978 Kühn-Rosendahl)

EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE JORDAN TRIPLE SYSTEM INTO A JORDAN TRIPLE MODULE IS INNER

(Lie algebras)

(i') ASSOCIATIVE TRIPLE SYSTEMS

derivation:

$$D(ab^tc) = ab^tDc + a(Db)^tc + (Da)b^tc$$

inner derivation: see Table 3

The (non-module) result can be derived from the result for Jordan triple systems. (See an exercise)

THEOREM (1976 Carlsson)
EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE ASSOCIATIVE TRIPLE SYSTEM INTO A MODULE IS INNER
(Lie algebras)

(ii') LIE ALGEBRAS

THEOREM (Zassenhaus)
(early 20th century)
EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE LIE ALGEBRA INTO ITSELF IS INNER

THEOREM (Hochschild 1942)
EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE LIE ALGEBRA INTO A MODULE IS INNER

(ii') LIE TRIPLE SYSTEMS

THEOREM (Lister 1952)
EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE LIE TRIPLE SYSTEM INTO ITSELF IS INNER

THEOREM (Harris 1961)
EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE LIE TRIPLE SYSTEM INTO A MODULE IS INNER

Table 1 $M_n(\mathbf{R})$ (ALGEBRAS)

associative	Lie	Jordan
$ab = a \times b$	$[a, b] = ab - ba$	$a \circ b = ab + ba$
Noeth,Wedd 1920	Zassenhaus 1930	Jacobson 1949
Hochschild 1942	Hochschild 1942	Jacobson 1951

Table 3 $M_{m,n}(\mathbf{R})$ (TRIPLE SYSTEMS)

associative triple	Lie triple	Jordan triple
ab^tc	$[[a, b], c]$	$ab^tc + cb^ta$
	Lister 1952	Meyberg 1972
Carlsson 1976	Harris 1961	Kühn-Rosendahl 1978

Ho-Peralta-Russo work on ternary weak amenability for C^*-algebras and JB^*-triples

(1 and part of 3 are proved in appendix A which follows; Appendix B defined Jordan triple module)

1. COMMUTATIVE C*-ALGEBRAS ARE TERNARY WEAKLY AMENABLE (TWA)

2. COMMUTATIVE JB*-TRIPLES ARE APPROXIMATELY WEAKLY AMENABLE

3. $B(H), K(H)$ ARE TWA IF AND ONLY IF FINITE DIMENSIONAL

4. CARTAN FACTORS OF TYPE $I_{m,n}$ OF FINITE RANK WITH $m \neq n$, AND OF TYPE IV ARE TWA IF AND ONLY IF FINITE DIMENSIONAL

APPENDIX A

We shall now prove that $\mathcal{C}$ and $M_n(\mathbf{C})$ are ternary weakly amenable and that $K(H)$ is not Jordan weakly amenable unless finite dimensional.

Our first results establish some technical connections between associative and ternary derivations from a *-algebra A to a Jordan A-module (resp., associative A-bimodule).

Given an algebra A, $a \in A$ and $\varphi \in A^*$, $a\varphi$, φa will denote the elements in A^* given by

$$a\varphi(y) = \varphi(ya) \quad and \quad \varphi a(y) = \varphi(ay), \ \ (y \in A).$$

LEMMA 1

Let A be an associative unital *-algebra (which we consider as a Jordan algebra), X a unital Jordan A-module and let $\delta : A_{sa} \to X$ be a (real) linear mapping. The following assertions are equivalent:

(a) δ is a ternary derivation and $\delta(1) = 0$.

(b) δ is a Jordan derivation.

PROOF

Since X is a unital real Jordan A_{sa}-module and $\delta(1) = 0$, the identity $\delta(a \circ b) = \delta\{a, 1, b\}$ $= \{\delta(a), 1, b\} + \{a, \delta(1), b\} + \{a, 1, \delta(b)\} = \{\delta(a), 1, b\} + \{a, 1, \delta(b)\} = \delta(a) \circ b + a \circ \delta(b)$, gives the implication 1. $\Rightarrow$ 2.

For every Jordan derivation $\delta : A_{sa} \to X$, we have $\delta(1) = \delta(1 \circ 1) = 2(1 \circ \delta(1)) = 2\delta(1)$, and hence $\delta(1) = 0$. The implication 2. $\Rightarrow$ 1. follows straightforwardly.

Henceforth, given a unital associative *-algebra, A, and a Jordan A-module, X, we shall write $\mathcal{D}_t^o(A,X)$ for the set of all ternary derivations from A to X vanishing at the unit element.

Given a *-algebra A, we consider the involution * on A^* defined by $\varphi^*(a) := \overline{\varphi(a^*)}$ ($a \in A$, $\varphi \in A^*$).

An element $\delta \in \mathcal{D}_J(A,A^*)$ is called a **-derivation* if $\delta(a^*) = \delta(a)^*$, for every $a \in A$.

The symbols $\mathcal{D}_J^*(A,A^*)$ and $\mathcal{I}nn_J^*(A,A^*)$ (resp., $\mathcal{D}_b^*(A,A^*)$ and $\mathcal{I}nn_b^*(A,A^*)$) will denote the sets of all Jordan and Jordan-inner (resp., associative and inner) *-derivations from A to A^*, respectively.

LEMMA 2

Let X be an A-bimodule over a *-algebra A. Then the following statements hold:

(a) $\mathcal{I}nn_J(A,X) \subset \mathcal{I}nn_b(A,X)$. In particular, $\mathcal{I}nn^*_J(A,A^*) \subset \mathcal{I}nn^*_b(A,A^*)$.

(b) Let D be an element in $\mathcal{I}nn_b(A,A^*)$, that is, $D = D_\varphi$† for some φ in A^*. Then D is a *-derivation whenever $\varphi^* = -\varphi$.

PROOF

(a): Let us consider a Jordan derivation of the form $\delta_{x_0,b}$, where $x_0 \in X$ and $b \in A$. For each a in A, we can easily check that $\delta_{x_0,b}(a) = (x_0 \circ a) \circ b - (b \circ a) \circ x_0 = \frac{1}{2}([b,x_0]a - a[b,x_0]) = D_{\frac{1}{2}[b,x_0]}(a)$, where the Lie bracket $[.,.]$ is defined by $[b,x_0] = \frac{1}{2}(bx_0 - x_0b)$ for every $b \in A$, $x_0 \in X$. Since every inner Jordan derivation D from A to X must be a finite sum of the form $D = \sum_{j=1}^n \delta_{x_j,b_j}$, with $x_j \in X$ and $b_j \in A$, it follows that $D = \sum_{j=1}^n D_{\frac{1}{2}[b_j,x_j]} = D_{\frac{1}{2}\sum_{j=1}^n [bj,x_j]}$ is an inner (associative) binary derivation.

† D_φ denotes the derivation $x \mapsto \varphi x - x\varphi$

(b) Let $D = D_\varphi$, where $\varphi \in A^*$ and $\varphi^* = -\varphi$. Let us fix two arbitrary elements a, b in A. The identities

$$D_\varphi(a^*)(b) = (\varphi a^* - a^*\varphi)\,(b) = \varphi(a^*b - ba^*)$$

and

$$D_\varphi(a)^*(b) = (\varphi a - a\varphi)^*\,(b) = (a^*\varphi^* - \varphi^*a^*)\,(b) = \varphi^*(ba^* - a^*b),$$

give $D_\varphi(a^*) = D_\varphi(a)^*$, proving that D is a *-derivation.

LEMMA 3

Let A be a unital *-algebra equipped with the ternary product given by $\{a,b,c\} = \frac{1}{2}(ab^*c + cb^*a)$. Every ternary derivation δ in $\mathcal{D}_t(A, A^*)$ satisfies the identity $\delta(1)^* = -\delta(1)$, that is, $\overline{\delta(1)(a^*)} = -\delta(1)(a)$, for every a in A.

PROOF

Let $\delta : A \to A^*$ be a ternary derivation. Since the identity

$\delta(1)(a) = \delta(\{1,1,1\})(a) = \{\delta(1),1,1\}(a) + \{1,\delta(1),1\}(a) + \{1,1,\delta(1)\}(a) = 2\delta(1)\{1,1,a\} + \overline{\delta(1)\{1,a,1\}} = 2\delta(1)(a) + \delta(1)^*(a)$,

holds for every $a \in A$, we do have $\delta(1)^* = -\delta(1)$.

LEMMA 4

Let A be a unital *-algebra equipped with the ternary product given by $abc = \frac{1}{2}\ (ab^*c + cb^*a)$. Then

$$\mathcal{D}_t(A, A^*) = \mathcal{D}_t^o(A, A^*) + \mathcal{I}nn_t(A, A^*).$$

More precisely, if $\delta \in \mathcal{D}_t(A, A^*)$, then $\delta = \delta_0 + \delta_1$, where $\delta_0 \in \mathcal{D}_t^o(A, A^*)$ and δ_1, defined by $\delta_1(a) := \delta(1) \circ a^* = \frac{1}{2}(\delta(1)\ a^* + a^*\ \delta(1))$, is the inner derivation $-\frac{1}{2}\delta(1, \delta(1))$.[‡]

PROOF

Let $\delta : A \to A^*$ be a ternary derivation. The mapping $\delta_1 : A \to A^*\ \delta_1(a) := \delta(1) \circ a^*$ is a conjugate-linear mapping with $\delta_1(1) = \delta(1)$. We will show that $\delta_1 = -\frac{1}{2}\delta(1, \delta(1))$. Then, the mapping $\delta_0 = \delta - \delta_1$ is a triple derivation with $\delta_0(1) = 0$ and $\delta = \delta_0 + \delta_1$, proving the lemma.

[‡] $\delta(a, \varphi)$ denotes the derivation $x \mapsto \{a\varphi x\} - \{\varphi a x\}$

Lemma 3 above implies that $\delta(1)^* = -\delta(1)$.

Now we consider the inner triple derivation $-\frac{1}{2}\delta(1,\delta(1))$. For each a and b in A we have

$$-\frac{1}{2}\delta(1,\delta(1))(a)(b) = -\frac{1}{2}\left(\{1\delta(1)a\} - \{\delta(1)1a\}\right)(b)$$

$$= -\frac{1}{2}\left(\overline{\delta(1)(\{1ba\})} - \delta(1)(\{1ab\})\right)$$

$$= -\frac{1}{2}\left(\delta(1)^*(\{1ab\}) - \delta(1)(\{1ab\})\right)$$

(since $\delta(1)^* = -\delta(1)$)
$= -\frac{1}{2}\left(-\delta(1)(\{1ab\}) - \delta(1)(\{1ab\}\right)$
$= \delta(1)(1ab) = \delta(1)(a^* \circ b) = \delta_1(a)(b)$.
Thus, $\delta_1 = -\frac{1}{2}\delta(1,\delta(1))$ as promised.

LEMMA 5

Let A be a unital *-algebra, let $D : A \to A^*$ be a linear mapping and let $\delta : A \to A^*$ denote the conjugate linear mapping defined by $\delta(a) := D(a^*)$. Then D lies in $\mathcal{D}_J(A, A^*)$ if, and only if, $\delta\{a1b\} = \{\delta(a)1b\} + \{a1\delta(b)\}$ for all $a, b \in A$. Moreover, $\mathcal{D}_t^o(A, A^*) = \{\delta : A \to A^* : \exists D \in \mathcal{D}_J^*(A, A^*)$
s.t. $\delta(a) := D(a^*), (a \in A)\}$.

PROOF

The first statement follows immediately from the definitions, that is, $\{\delta a1b\} = D(a^*) \circ b^*$, $\{a1\delta b\} = D(b^*) \circ a^*$, and $\delta\{a1b\} = D(a^* \circ b^*)$.

Suppose next that $\delta \in \mathcal{D}_t^o(A, A^*)$. From the first statement, D lies in $\mathcal{D}_J(A, A^*)$. Actually D is *-derivation; if $a \in A$ then $\delta(a^*) = \delta\{1a1\} = \{1\delta(a)1\}$, so for all $y \in A$, we have

$$\langle \delta(a^*), y \rangle = \langle \{1\delta(a)1\}, y \rangle = \overline{\langle \delta(a), \{1y1\} \rangle}$$

$= \langle (\delta(a))^*, y \rangle$, and hence
$D(a^*) = \delta(a) = (\delta(a^*))^* = (Da)^*$.

Suppose now that $D \in \mathcal{D}_J^*(A, A^*)$. It follows from the definitions and the fact that $D \in \mathcal{D}_J(A, A^*)$ that the following three equations hold:

$$\delta\{aba\} = 2(D(a^*) \circ b) \circ a^* + 2(a^* \circ D(b)) \circ a^*$$

$$+2(a^* \circ b) \circ D(a^*)$$

$$-2(D(a^*) \circ a^*) \circ b - (a^* \circ a^*) \circ D(b),$$

$$\{\delta(a)ba\} = D(a^*)\circ(b\circ a^*)+(D(a^*)\circ b)\circ a^*-(D(a^*)\circ a^*)\circ b$$

and

$$\{a\delta(b)a\} = 2((D(b^*))^* \circ a^*) \circ a^* - D(b) \circ (a^* \circ a^*).$$

From these three equations, we have

$$\delta\{aba\} - 2\{\delta(a)ba\} - \{a\delta(b)a\}$$

$$= 2(a^* \circ D(b)) \circ a^* - 2((D(b^*))^* \circ a^*) \circ a^*.$$

Since D is self-adjoint, the right side of the last equation vanishes, and the result follows.

PROPOSITION 1

Let A be a unital *-algebra. Then

$$\mathcal{D}_t(A, A^*) \subset \mathcal{D}_J^*(A, A^*) \circ * + \mathcal{I}nn_t(A, A^*).$$

If A is Jordan weakly amenable, then

$$\mathcal{D}_t(A, A^*) = \mathcal{I}nn_b^*(A, A^*) \circ * + \mathcal{I}nn_t(A, A^*),$$

i.e., $\mathcal{D}_t(A, A^*) = \{\delta(\psi, \varphi) : \psi, \varphi \in A^*\}$, where

$$\delta(\psi, \varphi)(a) = \psi a^* + a^* \varphi, \quad (a \in A).$$

PROOF

Let $\delta : A \to A^*$ be a ternary derivation. By Lemma 4, $\delta = \delta_0 + \delta_1$, where $\delta_0 \in \mathcal{D}_t^o(A, A^*)$, $\delta_1(a) = -\frac{1}{2}\delta(1, \delta(1))(a) = \delta(1) \circ a^*$. Lemmas 1 and 5 assure that $D = \delta_0 \circ *$, is a Jordan *-derivation. This proves the first statement.

The assumed Jordan weak amenability of A, together with Lemma 2 implies that $D = \delta_0 \circ *$ lies in $\mathcal{I}nn_b^*(A, A^*)$, which gives $\delta = D \circ * + \delta_1 \in \mathcal{I}nn_b^*(A, A^*) \circ * + \mathcal{I}nn_t(A, A^*)$. Since a simple calculation shows that $\mathcal{I}nn_b^*(A, A^*) \subset \mathcal{D}_t(A, A^*)$, the reverse inclusion holds, proving the second statement.

Since D is a binary inner derivation, there exists $\phi \in A^*$ such that $D = D_\phi$. Therefore

$$\delta(a) = D_\phi(a^*) + \delta(1) \circ a^* = \phi\, a^* - a^*\, \phi + \frac{\delta(1)}{2}\, a^* + a^*\, \frac{\delta(1)}{2}$$

$$= \left(\phi + \frac{\delta(1)}{2}\right)\, a^* - a^*\, \left(\frac{\delta(1)}{2} - \phi\right).$$

The final statement follows taking $\psi = \left(\phi + \frac{\delta(1)}{2}\right)$ and $\varphi = \left(\frac{\delta(1)}{2} - \phi\right)$.

When a *-algebra A is commutative, we have $\mathcal{I}nn_b(A, A^*) = \{0\}$. In the setting of unital and commutative *-algebras, the above Proposition implies the following:

COROLLARY 1

Let A be a unital and commutative *-algebra. Then A is ternary weakly amenable whenever it is Jordan weakly amenable.

Every C*-algebra A is binary weakly amenable (Haagerup 1983), and by Peralta and Russo 2010, every Jordan derivation $D : A \to A^*$ is continuous, and hence an associative derivation by Johnson's Theorem 1996. This gives us the next theorem.

THEOREM (Ho-Peralta-Russo)

Every unital and commutative (real or complex) C*-algebra is ternary weakly amenable.

We next present an example of a C*-algebra which is not ternary weakly amenable.

LEMMA 6

The C*-algebra $A = K(H)$ of all compact operators on an infinite dimensional Hilbert space H is not Jordan weakly amenable.

PROOF

By the theorems of Johnson and Haagerup referred to several times already, we have

$$\mathcal{D}_J(A, A^*) = \mathcal{D}_b(A, A^*) = \mathcal{I}nn_b(A, A^*).$$

We shall identify A^* with the trace-class operators on H.

Supposing that A were Jordan weakly amenable, let $\psi \in A^*$ be arbitrary. Then D_ψ would be an inner Jordan derivation, so there would exist $\varphi_j \in A^*$ and $b_j \in A$ such that $D_\psi(x) = \sum_{j=1}^n [\varphi_j \circ (b_j \circ x) - b_j \circ (\varphi_j \circ x)]$ for all $x \in A$.

For $x, y \in A$, a direct calculation yields

$$\psi(xy - yx) = -\frac{1}{4} \left(\sum_{j=1}^n b_j \varphi_j - \varphi_j b_j \right) (xy - yx).$$

It is known since 1971 (Pearcy and Topping) that every compact operator on a separable infinite dimensional Hilbert space is a finite sum of commutators of compact operators. Let z be any element in $A = K(H)$. Thus, z can be written as a finite sum of commutators $[x, y] = xy - yx$ of elements x, y in $K(H)$. Thus, it follows that the trace-class operator $\psi = -\frac{1}{4}\left(\sum_{j=1}^{n} b_j\varphi_j - \varphi_j b_j\right)$ is a finite sum of commutators of compact and trace-class operators, and hence has trace zero. This is a contradiction, since ψ was arbitrary.

PROPOSITION 2

The C*-algebra $A = K(H)$ of all compact operators on an infinite dimensional Hilbert space H is not ternary weakly amenable.

PROOF

Let ψ be an arbitrary element in A^*. The binary inner derivation $D_\psi : x \mapsto \psi x - x\psi$ may be viewed as a map from either A or A^{**} into A^*. Considered as a map on A^{**}, it belongs to $\mathcal{I}nn_b(A^{**}, A^*)$ so by a technical Corollary to Proposition 1, $D_\psi \circ * : a \mapsto D_\psi(a^*)$, belongs to $\mathcal{D}_t(A^{**}, A^*)$.

Assuming that A is ternary weakly amenable, the restriction of $D_\psi \circ *$ to A belongs to $\mathcal{I}nn_t(A, A^*)$. Thus, there exist $\varphi_j \in A^*$ and $b_j \in A$ such that $D_\psi \circ * = \sum_{j=1}^n (L(\varphi_j, b_j) - L(b_j, \varphi_j))$ on A.

For $x, a \in A$, direct calculations yield

$$\psi(a^*x - xa^*) =$$

$$\frac{1}{2}\sum_{j=1}^n (\varphi_j b_j - b_j^* \varphi_j^*)(a^*x) + \frac{1}{2}\sum_{j=1}^n (b_j \varphi_j - \varphi_j^* b_j^*)(xa^*).$$

We can and do set $x = 1$ to get

$$0 = \frac{1}{2}\sum_{j=1}^{n}(\varphi_j b_j - b_j^* \varphi_j^*)(a^*) + \frac{1}{2}\sum_{j=1}^{n}(b_j\varphi_j - \varphi_j^* b_j^*)(a^*), \tag{3}$$

and therefore

$$\psi(a^*x - xa^*) = \frac{1}{2}\sum_{j=1}^{n}(\varphi_j b_j - b_j^*\varphi_j^*)(a^*x - xa^*), \tag{4}$$

for every $a, x \in A$.

Using the 1971 result as in the proof of Lemma 6, and taking note of (4) and (3), we have

$$\psi = \frac{1}{2}\sum_{j=1}^{n}(\varphi_j b_j - b_j^*\varphi_j^*) = \frac{1}{2}\sum_{j=1}^{n}(\varphi_j^* b_j^* - b_j\varphi_j).$$

Hence

$$\begin{aligned} 2\psi &= \sum_{j=1}^{n}(\varphi_j b_j - b_j\varphi_j + b_j\varphi_j - \varphi_j^* b_j^* + \varphi_j^* b_j^* - b_j^*\varphi_j^*) \\ &= \sum_{j=1}^{n}[\varphi_j, b_j] - 2\psi + \sum_{j=1}^{n}[\varphi_j^*, b_j^*]. \end{aligned}$$

Finally, the argument given at the end of the proof of Lemma 6 shows that ψ has trace 0, which is a contradiction, since ψ was arbitrary.

LEMMA 7

Let A denote the JB*-triple $M_n(\mathbf{C})$. Then

$$\mathcal{I}nn_b^*(A,A^*) = \mathcal{I}nn_J^*(A,A^*) = \mathcal{D}_J^*(A,A^*)$$

PROOF

Let $D \in \mathcal{I}nn_b^*(A,A^*)$ so that $D(x) = \psi x - x\psi$ for some $\psi \in A^*$. Recall (1971) that every compact operator is a finite sum of commutators of compact operators. Therefore, by Lemma 2, $\psi^* = -\psi$. Also, since every matrix of trace 0 is a commutator (1937 Shoda,1957 Albert&Muckenhoupt), we have

$$\psi = [\varphi, b] + \frac{\mathrm{Tr}\,(\psi)}{n} I.$$

Expanding $\varphi = \varphi_1 + i\varphi_2$ and $b = b_1 + ib_2$ into hermitian and skew symmetric parts and using $\psi^* = -\psi$ leads to

$$\psi = [\varphi_1, b_1] - [\varphi_2, b_2] + \frac{\mathrm{Tr}\,(\psi)}{n} I.$$

For $x, y \in A$, direct calculation yields

$$D(x) = \varphi_1\circ(b_1\circ x) - b_1\circ(\varphi_1\circ x) - \varphi_2\circ(b_1\circ x) + b_2\circ(\varphi_2\circ x),$$

so that $D \in \mathcal{I}nn_J^*(A, A^*)$.

From this, and the theorems of Johnson and Haagerup, we have $\mathcal{D}_J^*(A, A^*) = \mathcal{D}_b^*(A, A^*) = \mathcal{I}nn_b^*(A, A^*) \subseteq \mathcal{I}nn_J^*(A, A^*) \subseteq \mathcal{D}_J^*(A, A^*)$.

PROPOSITION 3

The JB*-triple $A = M_n(\mathbf{C})$ is ternary weakly amenable.

PROOF

By Proposition 1, $\mathcal{D}_t(A, A^*) = \mathcal{I}nn_b^*(A, A^*) \circ * + \mathcal{I}nn_t(A, A^*)$, so it suffices to prove that $\mathcal{I}nn_b^*(A, A^*) \circ * \subset \mathcal{I}nn_t(A, A^*)$.

As in the proof of Lemma 7, if $D \in \mathcal{I}nn_b^*(A, A^*)$ so that $Dx = \psi x - x\psi$ for some $\psi \in A^*$, then

$$\psi = [\varphi_1, b_1] - [\varphi_2, b_2] + \frac{\mathrm{Tr}\,(\psi)}{n} I,$$

where b_1, b_2 are self adjoint elements of A and φ_1 and φ_2 are self adjoint elements of A^*.

It is easy to see that, for each $x \in A$, we have $D(x^*) = \{\varphi_1, 2b_1, x\} - \{2b_1, \varphi_1, x\}$ $-\{\varphi_2, 2b_2, x\} + \{2b_2, \varphi_2, x\}$, so that $D \circ * \in \mathcal{I}nn_t(A, A^*)$.

Appendix B: Jordan triple modules

If A is an associative algebra, an A-*bimodule* is a vector space X, equipped with two bilinear products $(a,x) \mapsto ax$ and $(a,x) \mapsto xa$ from $A \times X$ to X satisfying the following axioms:

$$a(bx) = (ab)x, \;\; a(xb) = (ax)b, \text{ and, } (xa)b = x(ab),$$

for every $a, b \in A$ and $x \in X$.

If J is a Jordan algebra, a *Jordan J-module* is a vector space X, equipped with two bilinear products $(a,x) \mapsto a \circ x$ and $(x,a) \mapsto x \circ a$ from $J \times X$ to X, satisfying:

$$a \circ x = x \circ a, \quad a^2 \circ (x \circ a) = (a^2 \circ x) \circ a, \text{ and,}$$

$$2((x \circ a) \circ b) \circ a + x \circ (a^2 \circ b) = 2(x \circ a) \circ (a \circ b) + (x \circ b) \circ a^2,$$

for every $a, b \in J$ and $x \in X$

If E is a complex Jordan triple, a *Jordan triple* *E-module* (also called *triple E-module*) is a vector space X equipped with three mappings

$\{.,.,.\}_1 : X \times E \times E \to X$
$\{.,.,.\}_2 : E \times X \times E \to X$
$\{.,.,.\}_3 : E \times E \times X \to X$

satisfying:

1. $\{x,a,b\}_1$ is linear in a and x and conjugate linear in b, $\{abx\}_3$ is linear in b and x and conjugate linear in a and $\{a,x,b\}_2$ is conjugate linear in a,b,x
2. $\{x,b,a\}_1 = \{a,b,x\}_3$, and $\{a,x,b\}_2 = \{b,x,a\}_2$ for every $a,b \in E$ and $x \in X$.
3. Denoting by ... any of the products $\{.,.,.\}_1$, $\{.,.,.\}_2$ and $\{.,.,.\}_3$, the identity $abcde = abcde - cbade + cdabe$, holds whenever one of the elements a,b,c,d,e is in X and the rest are in E.

It is a little bit laborious to check that the dual space, E^*, of a complex (resp., real) Jordan Banach triple E is a complex (resp., real) triple E-module with respect to the products:

$$\{ab\varphi\}(x) = \{\varphi ba\}(x) := \varphi\{bax\} \quad (5)$$

and

$$\{a\varphi b\}(x) := \overline{\varphi\{axb\}}, \quad (6)$$

for all $\varphi \in E^*, a, b, x \in E$.