Complex Analysis Math 147—Winter 2008
Solutions to Review Problems for final exam; March 14,2008

1. Let f be entire and suppose that the second derivative of f is bounded: $|f^{''}(z)| \leq M$ for all $z \in \mathbf{C}$. Prove that f is a polynomial of degree at most 2.

   Hint: Use Liouville's theorem on f'' and then the fact that in a polygonally connected open set, an analytic function whose derivative vanishes everywhere must be a constant.

   Solution: f'' is entire, so by Liouville's theorem, it is a constant, say $f''(z) = c \in \mathbf{C}$ for all $z \in \mathbf{C}$. Let $g(z) = cz$. Then $g'(z) = f''(z)$ for all z and since $\mathbf{C}$ is connected, $f'(z) = g(z)+d$ where d is a complex constant. Now let $h(z) = cz^2/2+dz$. Then $h'(z) = cz+d = f'(z)$, so $f(z) = h(z)+e$ where e is a complex constant. Thus $f(z) = cz^2/2 + dz + e$ is a polynomial of degree at most two. $\square$

2. Suppose that f is entire and that $|f(z)| \leq |z|^{2.5}$ for all $|z| > 100$.
   (a) Given z_0, show that for R sufficiently large, $|f^{(3)}(z_0)| \leq 3!R^{3.5}/(R - |z_0|)^4)$
   (b) Prove that f must be a polynomial of degree at most 2.

   Hint: Use the Cauchy integral formula for the third derivative of f and estimate the integral using the fact that since $|z| = R$, $|z - z_0| \geq R - |z_0|$

   Solution: For any $R > |z_0|$, by Cauchy's integral formula, $f'''(z_0) = \frac{3!}{2\pi i}\int_{|z|=R} \frac{f(z)}{(z-z_0)^4}\,dz$ so that, since for $|z| = R$, $|z - z_0| \geq R - |z_0|$, $|f'''(z_0)| \leq \frac{3}{\pi}\frac{R^{2.5}}{(R-|z_0|)^4} \cdot 2\pi R \to 0$ as $R \to \infty$. Thus $f'''(z) = 0$ for all z, and f'' is a constant. By Problem 1, f is a polynomial of degree at most 2. $\square$

3. Let f be an entire function and let $a, b \in \mathbf{C}$.
   (a) Evaluate the integral $\int_{|z|=R} \frac{f(z)}{(z-a)(z-b)}\,dz$ if $R > |a|$ and $R > |b|$.
   (b) Use the result of (a) to give a proof of Liouville's theorem.

   Hint: Estimate the integral and let $R \to \infty$.

   Solution: By the main application of Cauchy's homotopy theorem,

$$\int_{|z|=R} \frac{f(z)}{(z-a)(z-b)}\,dz = \int_{|z-a|=R_a} \frac{f(z)}{(z-a)(z-b)}\,dz + \int_{|z-b|=R_b} \frac{f(z)}{(z-a)(z-b)}\,dz$$

   for suitable radii R_a and R_b. In turn, these integrals are given by Cauchy's integral formula to obtain

$$\int_{|z|=R} \frac{f(z)}{(z-a)(z-b)}\,dz = 2\pi i\left[\frac{f(a)}{a-b} + \frac{f(b)}{b-a}\right]. \tag{1}$$

   This answers (a).
   If f is bounded, say $|f(z)| \leq M$ for all $z \in \mathbf{C}$, then for all $|z| = R$,

$$\left|\frac{f(z)}{(z-a)(z-b)}\right| \leq \frac{M}{(R-|a|)(R-|b|)}$$

   so that, estimating the integral in (1),

$$\left|\frac{f(a)-f(b)}{a-b}\right| \leq \frac{2\pi M}{(R-|a|)(R-|b|)} \cdot 2\pi R \to 0$$

   as $R \to \infty$. This proves that $f(a) = f(b)$ for any two points a and b, so f is a constant.

4. (a) Show that the following function is analytic on the open unit disk:$f(z) = \int_0^1 \frac{dt}{1-tz}$

   Hint: Use Morera's theorem and an interchange of the order of integration.

   (b) Find a power series expansion for this function.

   Hint: Use the known power series for the integrand and interchange the summation and integration.

   Solution: Let T be a triangle in $|z| < 1$. Then[1]

$$\int_T f(z)\,dz = \int_T \left(\int_0^1 \frac{1}{1-tz}\,dt \right) dz = \int_0^1 \left(\int_T \frac{1}{1-tz}\,dz \right) dt.$$

   The inner integral is zero since for a fixed t, $1/(1-tz)$ is an analytic function of z. Since f is continuous, Morera's theorem applies and f is analytic.

   Integrating term by term[2] we get

$$f(z) = \int_0^1 \left(\sum_0^\infty t^n z^n \right) dt = \sum_0^\infty \left(\int_0^1 t^n\,dt \right) z^n = \sum_1^\infty z^n/n.$$

5. Let $0 < r < R$ and $A := \{z \in \mathbf{C} : r \le |z| \le R\}$. Show that there is a positive number ϵ such that for each polynomial p.

$$\sup\{|p(z) - z^{-1}| : z \in A\} \ge \epsilon.$$

   Hint: Supposing it is not true, either find a sequence of polynomials converging uniformly to $1/z$ on compact subsets of A, then integrate over a circle, or alternatively, use the estimate on contour integrals.

   Solution: Assuming the assertion is false, take $\epsilon = 1/n$ to get a polynomial p_n such that

$$\sup\{|p_n(z) - z^{-1}| : z \in A\} < 1/n.$$

   This implies that p_n converges uniformly on A to $1/z$. Let C be the circle of radius $(r+R)/2$. Since p_n is analytic on and inside C, $\int_C p_n(z)\,dz = 0$. Since $p_n \to 1/z$ uniformly on C, $\int_C p_n(z)\,dz \to \int_C \frac{1}{z}\,dz$, which says that $\int_C \frac{1}{z}\,dz = 0$. This is a contradiction, since $\int_C \frac{1}{z}\,dz = 2\pi i$.

6. Let f_n be a sequence of functions which are continuous on the closed unit disk $\{|z| \le 1\}$ and analytic on the open disk $\{|z| < 1\}$. Suppose the f_n converges uniformly to a function f on the unit circle $\{|z| = 1\}$. Show that f can be extended to a function g on $\{|z| \le 1\}$ which is analytic on $\{|z| < 1\}$.

   Hint: Apply the maximum modulus theorem to $f_n - f_m$.

   Solution: The sequence f_n is uniformly Cauchy on $|z| = 1$. So for $\epsilon > 0$ $\exists N$ such that $\max_{|z|=1} |f_n(z) - f_m(z)| < \epsilon$ whenever $m, n > N$. By the Maximum Modulus Theorem, $\max_{|z|\le 1} |f_n(z) - f_m(z)| < \epsilon$ whenever $m, n > N$. Thus for each $|z| \le 1$, $f_n(z)$ is a Cauchy sequence, so it converges to a number, call it $g(z)$, and g is an extension of f to $|z| \le 1$. In fact, f_n converges uniformly to g on $|z| \le 1$. Indeed, for a fixed z, $|f_n(z) - f_m(z)| < \epsilon$ whenever $m, n > N$, so that letting $n \to \infty$, $|g(z) - f_m(z)| < \epsilon$ whenever $m > N$. By Proposition 21.1, g is analytic on $|z| < 1$.□

[1] You really should justify the interchange of the order of integration

[2] This can be justified using uniform convergence of the series on T

7. (a) Prove that if f is an automorphism of the open unit disk (that is, $f : \{|z| < 1\} \to \{|z| < 1\}$ is analytic, one-to-one and onto) and if $f(0) = 0$, then $f(z) = e^{i\theta} z$ for all $|z| < 1$

Hint: Assuming, as you may, that f^{-1} is analytic, use Schwarz's lemma for f and f^{-1}.

Solution: By Schwarz's lemma for f and f^{-1},

$$|f(z)| \le |z| \text{ and } |f^{-1}(z)| \le |z| \text{ for every } |z| < 1.$$

In particular, for some $z_0 \neq 0$, $|f(z_0)| = |z_0|$, so by part (b) of Schwarz's lemma, $f(z) = cz$ with $|c| = 1$.

For (b) and (c), let D be an open subset of $\mathbf{C}$ and fix a point $a \in D$.

(b) Show that there is at most one analytic function $f : D \to \{|z| < 1\}$ which is one-to-one and onto and satisfies $f(a) = 0$ and $f'(a) > 0$.

Hint: If f and g are two such functions, then $f \circ g^{-1}$ is an automorphism of the open unit disk. (Also, recall that $(g^{-1})'(g(z)) = 1/g'(z)$).

Solution: Since $f \circ g^{-1}(0) = 0$, by (a), $f \circ g^{-1}(z) = e^{i\theta} z$. Taking the derivative, we obtain $e^{i\theta} = f'(g^{-1}(z))(g^{-1})'(z)$ and in particular, $e^{i\theta} = f'(g^{-1}(0))(g^{-1})'(0) = f'(a)\frac{1}{g'(a)} > 0$. Therefore $e^{i\theta} = 1$, and $f = g$.

(c) Let g be any one-to-one analytic function mapping D onto the open unit disk. Show that

$$g(z) = \frac{e^{i\theta} f(z) + \alpha}{1 + \overline{\alpha} e^{i\theta} f(z)} \qquad (z \in D)$$

where[3] α and θ are defined by $\alpha = g(a)$ and $g'(a) = e^{i\theta}|g'(a)|$.

Hint: Let $h := \varphi_{-\alpha} \circ g$ and evaluate $h'(a)$, where as usual $\varphi_\alpha(z) = \frac{z+\alpha}{1+\overline{\alpha} z}$

Solution: Note first that $h(a) = \varphi_{-\alpha}(g(a)) = 0$. Next note that by direct calculation, $(\varphi_{-\alpha})'(\alpha) = 1/(1 - |\alpha|^2)$. Hence $h'(a) = (\varphi_{-\alpha})'(g(a))g'(a) = (1 - |\alpha|^2)^{-1} e^{i\theta} |g'(a)|$.

It follows that $(e^{-i\theta} h)(a) = 0$ and $(e^{-i\theta} h)'(a) = \frac{|g'(a)|}{1-|\alpha|^2} > 0$, so by part (b), $(e^{i\theta} h)(z) = f(z)$, or

$$g(z) = (\varphi_\alpha \circ h)(z) = \varphi_\alpha(e^{i\theta} f(z)) = \frac{e^{i\theta} f(z) + \alpha}{1 + \overline{\alpha} e^{i\theta} f(z)}.$$

8. Let f be analytic on an open polygonally connected set D containing an open interval I of the real axis. Suppose that D is symmetric about the real axis: $z \in D \Leftrightarrow \overline{z} \in D$, and that f is real-valued on I. Prove that $\overline{f(z)} = f(\overline{z})$ for all $z \in D$.

Hint: Define $g(z) = \overline{f(\overline{z})}$. Show, by using the Cauchy-Riemann equations that g is analytic on D. Then use the identity theorem to show that $f = g$.

Solution: If $f = u + iv$ and $g = U + iV$, then by definition of g, $U(x, y) = u(x, -y)$ and $V(x, y) = -v(x, -y)$, so that $U_x = u_x$, $U_y = -u_y$, $V_x = -v_x$ and $V_y = v_y$. Therefore, $U_x = u_x = v_y = V_y$ and $U_y = -u_y = v_x = -V_x$. Thus the Cauchy-Riemann equations hold for U and V and g is analytic. Since $g = f$ on I, which has a limit point in D, by the identity theorem, $f = g$ on all of D.

[3] f is the unique (if it exists!) analytic one-to-one function of D onto the open unit disk satisfying $f(a) = 0$ and $f'(a) > 0$ given by (b)

9. Show that there does not exist a one-to-one analytic function of $\{z \in \mathbf{C} : 0 < |z| < 1\}$ onto the open unit disk $\{z \in \mathbf{C} : |z| < 1\}$.

   Hint and **Solution**: If f is such a function, show first that it would have an analytic extension g to $z = 0$. **Since f is bounded the extension exists by Riemann's removable singularity theorem.** Then consider the cases $|g(0)| = 1$ and $|g(0)| < 1$. **In the former case, by the maximum modulus theorem, g would be a constant, clearly impossible since f is one-to-one.** In the latter case there exists β with $0 < |\beta| < 1$ and $f(\beta) = g(0)$. (**Since f is onto.**) (Note that $f(\beta) = g(\beta)$.) Obtain a contradiction by justifying the following steps:

   - For $\epsilon > 0$, there exists $\delta > 0$ such that $f^{-1}(B(g(0), \delta)) \subset B(\beta, \epsilon)$ (**Since f^{-1} is continuous at $f(\beta)$.**)
   - For this δ, there exists $\delta' > 0$ such that $g(B(0, \delta')) \subset B(g(\beta), \delta)$ and hence $f(B(0, \delta') - \{0\}) \subset B(g(\beta), \delta)$ (**Since g is continuous at 0.**)
   - $B(0, \delta') - \{0\} \subset B(\beta, \epsilon)$ (**This follows from the previous two statements and is the desired contradiction**.)

10. Suppose f is analytic and zero-free in a simply connected domain D. Show there is an analytic function g in D with $e^{g(z)} = f(z)$.

    Hint and **Solution:** Fix $z_0 \in D$ and define $h(z) = \int_{z_0}^{z} \frac{f'(w)}{f(w)}\, dw$ where the integral is taken over any curve in D from z_0 to z. Explain why h is well-defined (**Since D is simply connected, the integral is path independent.**) and analytic in D with derivative $h' = f'/f$ (**This is how antiderivatives were constructed.**) and then show that $\frac{d}{dz}(e^{-h(z)} f(z)) = 0$ (**By the product rule.**). **Finally, $e^{-h(z)} f(z) = c$, a constant which is not zero (since f is "zero free"). Writing $c = e^d$ for some other constant d, we have $f(z) = ce^{h(z)} = e^{dh(z)}$.**