Complex Analysis Math 147—Winter 2006
Solutions to Review Problems—Chapters 1-3; February 2, 2006

1. Write the number in the form $a + bi$ with $a, b \in \mathbf{R}$.

   $\frac{3}{i} + \frac{i}{3} = -8i/3 \quad , \quad (2+i)(-1-i)(3-2i) = -1-3i.$

2. Let z be a complex number such that $\Re z > 0$. Prove that $\Re(1/z) > 0$.

   $1/z = 1/(x+iy) = (x-iy)/(x^2+y^2)$ so $\mathrm{Re}\,(1/z) = x/(x^2+y^2) > 0$.

3. Evaluate $\qquad 3i^{11} + 6i^3 + \frac{8}{i^{20}} + i^{-1} = 8 - 10i$

4. Describe the set of points in the complex plane that satisfies each of the following

   $\Im z = 2$ is the line $y = 2$

   $|2z - i| = 4$ is the circle with center $i/2$ and radius 2

   $|z| = \Re z + 2$ is the parabola $y^2 = 4(x+1)$

   $|z| = 3|z-1|$ is the circle with center 9/8 and radius 3/8

   $|z - i| < 2$ is the inside of a circle with center i and radius 2.

5. Prove that if $(\overline{z})^2 = z^2$, then z is either real or purely imaginary.

   $(x-iy)^2 = (x+iy)2 \Rightarrow x^2 - 2ixy - y^2 = x^2 + 2ixy - y^2 \Rightarrow xy = 0$

6. Decide which of the following statements are true.

   (a) $\mathrm{Arg}\, z_1 z_2 = \mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2$ if $z_1 \neq 0$ and $z_2 \neq 0$. False

   (b) $\mathrm{Arg}\, \overline{z} = -\mathrm{Arg}\, z$ if z is not a real number. True

7. Write each of the given numbers in the form $a + bi$ with $a, b \in \mathbf{R}$.

   $\exp(-i\pi/4) = (1/\sqrt{2})(1-i) \quad , \quad \frac{\exp(1+i3\pi)}{\exp(-1+i\pi/2)} = e^2 i$

   $\exp(\exp i) = e^{\cos 1}(\cos(\sin 1) + i \sin(\sin 1))$

8. Write each of the given numbers in the polar form $r\cos\theta + ir\sin\theta$.

   $\left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right)^3 = \exp(i2\pi/3)$ or $\exp(i2\pi/3 + 2\pi k),\ k \in \mathbf{Z}$

   $\frac{2+2i}{-\sqrt{3}+i} = 2\sqrt{2}\exp(i\pi/4)$ or $2\sqrt{2}\exp(i\pi/4 + 2\pi k),\ k \in \mathbf{Z}$

   $\frac{2i}{3\exp(4+i)} = (2/3e^4)\exp(-i\pi/2)$ or $(2/3e^4)\exp(-i\pi/2 + 2\pi k),\ k \in \mathbf{Z}$

9. The function $\gamma(t) = \exp(it)$, $0 \le t \le 2\pi$ describes the unit circle traversed in the counter-clockwise direction. Describe each of the following curves.

   (a) $\gamma(t) = 3\exp(it)$, $0 \le t \le 2\pi$ is a circle with center 0 and radius 3 traversed in the counterclockwise direction once.

   (b) $\gamma(t) = 2\exp(it) + i$, $0 \le t \le 2\pi$ is a circle with center i and radius 2 traversed in the counterclockwise direction once.

10. Use the identity $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$ for any positive integer n to show that $(\sqrt{3} - i)^7 = -64\sqrt{3} + i64$.

$(\sqrt{3} - i)^7 = (2\exp(-i\pi/6)^7 = 2^7\exp(-7\pi/6) = 2^7(\sqrt{3}/2 + i/2) = -64\sqrt{3} + 64i$

11. Write each of the following functions in the form $w = u(x,y) + iv(x,y)$.

(a) $f(z) = 3z^2 + 5z + i + 1 = 3(x^2 - y^2) + 5x + 1 + i(6xy + 5y + 1)$

(b) $h(z) = \frac{z+i}{z^2+1} = x/(x^2 + (y-1)^2) + i(1-y)/(x^2 + (y-1)^2)$

(c) $F(z) = \exp(3z) = e^{3x}\cos 3y + ie^3x\sin 3y$

12. Show that the function $\operatorname{Arg} z$ is discontinuous at each point of the non-positive real axis.

First of all, the function $\operatorname{Arg} z$ is not defined for $z = 0$. Let $z_0 = x_0$ be a negative real number. If $y > 0$, then $\operatorname{Arg}(x_0 + iy) = \pi - \tan^{-1}(y/ - x_0) \to \pi$ as $y \to 0$.

Also, if $y < 0$, then $\operatorname{Arg}(x_0 + iy) = -\pi - \tan^{-1}(y/x_0) \to -\pi$ as $y \to 0$. Therefore, $\lim_{z\to x_0} \operatorname{Arg} z$ does not exist, and so $\operatorname{Arg} z$ is not continuous at $z_0 = x_0$ if $x_0 < 0$.

13. Prove that if f is continuous at $z = z_0$, then so are the functions $\overline{f(z)}$ and $\Re f(z)$.

$|\overline{f(z)} - \overline{f(z_0)}| = |f(z) - f(z_0)| < \epsilon$ if $|z - z_0| < \delta$

$|\operatorname{Re} f(z) - \operatorname{Re} f(z_0)| = |(f(z) + \overline{f(z)})/2 - (f(z_0) + \overline{f(z_0)})/2|$

$\leq \frac{1}{2}|f(z) - f(z_0)| + \frac{1}{2}|\overline{f(z)} - \overline{f(z_0)}|$

14. Find each of the following limits.

(a) $\lim_{z\to 2\pi i}(\exp z - \exp(-z)) = 0$

(b) $\lim_{z\to -\pi i}(z+1)\exp\left(\frac{z^2+\pi^2}{z+\pi i}\right) = 1 - \pi i$

15. Where are each of the following functions analytic?

(a) $8\overline{z} + i$ NOWHERE

(b) $\frac{z^3+2z+i}{z-5}$ ON $\mathbf{C} - \{5\}$

(c) $x^2 + y^2 + y - 2 + ix$ NOWHERE

(d) $|z|^2 + 2z$ NOWHERE

16. Use the Cauchy-Riemann equations to determine where the following functions are differentiable.

(a) $f(x+iy) = (x^{4/3}y^{5/3} + ix^{5/3}y^{4/3})/(x^2 + y^2)$ for $z \neq 0$ and $f(0) = 0$. THIS PROBLEM IS TOO MESSY—PLEASE IGNORE IT!

(b) $g(x+iy) = 3x^2 + 2x - 3y^2 - 1 + i(6xy + 2y)$. EVERYWHERE

17. Show that if f is analytic and real-valued on a connected open set, then f is a constant function.

    $f = u + iv$ and $v = 0$, so $u_x = v_y = 0$ and $u_y = v_x = 0$. Since we are on an open connected set, we have u is a constant and so f is constant.

18. Suppose $f(z)$ and $\overline{f(z)}$ are analytic on an open connected set. Show that f is a constant function.

    $f = u + iv$ analytic implies $u_x = v_y$ and $u_y = -v_x$. $\overline{f} = u - iv$ analytic implies $u_x = -v_y$ and $u_y = -(-v_x) = v_x$. Then $u_x = u_y = 0$ and u is a constant since the set is open and connected. Same for v.

19. Write each of the following numbers in the form $a + bi$.

    (a) $\exp(2 + i\pi/4) = e^2/\sqrt{2}(1 + i)$

    (b) $\cos(1 - i) = \cos 1 \cosh 1 + i \sin 1 \sinh 1$

20. Show that the formula $\exp(iz) = \cos z + i \sin z$ holds for all complex z.

    $\exp(iz) = \exp(-y + ix) = e^{-y} \cos x + ie^{-y} \sin x$ and

    $\cos z + i \sin z = \cos x \cosh y - i \sin x \sinh y + i \sin x \cosh y - \cos x \sinh y$

    $= \cos x(\cosh y - \sinh y) + i \sin x(\cosh y - \sinh y) = (\cos x)e^{-y} + i(\sin x)e^{-y}$

21. Evaluate each of the following.

    (a) $\log i = \{i(\pi/2 + 2\pi k) : k \in \mathbf{Z}\}$

    (b) $\text{Log}\,(\sqrt{3} + i) = \log \sqrt{2} + i\pi/6$

22. Find all values of the following.

    (a) $2^{\pi i} = \{\exp(\pi i \text{Log}\, 2 - 2\pi^2 k) : k \in \mathbf{Z}\}$

    (b) $(1 + i)^3 = 2(i - 1)$

23. Find the principal value of the following.

    (a) $i^{2i} = e^{-\pi}$

    (b) $(1 + i)^{1-i} = \sqrt{2}e^{\pi/4} \exp(i(\pi/4 - \log \sqrt{2}))$

24. Find all solutions of $\sin z = 2$.

    If $z = x + iy$ then $\sin z = 2$ if and only if $\sin x \cosh y = 2$ and $\cos x \sinh y = 0$. This leads to $z = \pi/2 + 2k\pi + i \cosh^{-1}(2)$ for any $k \in \mathbf{Z}$.