Elementary Analysis Math 140B—Winter 2007
Sample Midterm; January 31, 2007

1. (25%) Let $f_n(x) = 1/(nx+1)$ if $0 < x < \infty$, $n = 1, 2, \ldots$.
   - Does f_n converge uniformly on $(0, 1)$? Justify your answer.
   - Does f_n converge uniformly on $(1, \infty)$? Justify your answer.

2. (25%) Let $f_n(x) = |\sin \pi x|^{1/n}$ if $0 < x < \infty$, $n = 1, 2, \ldots$. (By definition, $0^{1/n} = 0$.)
   - Does f_n converge uniformly on $(0, 1)$? Justify your answer.
   - Does f_n converge uniformly on $(1, \infty)$? Justify your answer.

3. (25%) Consider the power series $\sum_{n=1}^{\infty} \sqrt{n} x^n$.
   - What is the interval of convergence of this series?
   - Show that the series does not converge uniformly on its interval of convergence.
     Hint: Use the fact that a series $\sum_n g_n(x)$ converges uniformly on a set S, then
     $$\lim_{n\to\infty} [\sup_{x\in S} |g_n(x)|] = 0.$$

4. (15%) Assume that f_n converges uniformly to f on S and suppose there is a constant $M > 0$ such that $|f_n(x)| \leq M$ for all $x \in S$ and all $n \geq 1$. Let g be continuous on $[-M, M]$.
   - Show that $|f(x)| \leq M$ for all $x \in S$.
   - Show that $g \circ f_n$ converges uniformly to $g \circ f$ on S.
     Hint: Use the uniform continuity of g on $[-M, M]$.

5. (10%) Suppose that f is a continuous function on $[0, 1]$ such that $\int_0^1 f(t)t^n\,dt = 0$ for $n = 0, 1, 2, \ldots$.

   Show that $\int_0^1 f(t)g(t)\,dt = 0$ for every continuous function g on $[0, 1]$.

   Hint: Approximate g uniformly by polynomials.