VI. Lie Triple Systems

6.1. Let $\mathcal{T}$ be a vector space over $\mathbb{R}$ or $\mathbb{C}$ ~~unital Φ-module.~~ $\mathcal{T}$ together with a trilinear map $(x,y,z) \longmapsto [xyz]$, is called a <u>Lie triple system</u> (= L.t.s.), if
$F \times F \times F \to F$

(i) $[xxz] = 0$

(6.1) (ii) $[xyz] + [yzx] + [zxy] = 0$ (Jacobi identity)

(iii) $[uv[xyz]] = [[uvx]yz] + [x[uvy]z] + [xy[uvz]]$

for all $u, v, x, y, z \in \mathcal{T}$.

Examples. 1) Let $\mathcal{L}$ be a Lie algebra with product $(x,y) \longmapsto [xy]$, then $\mathcal{L}$ together with $(x,y,z) \longmapsto [[xy]z]$ is a L.t.s.

2) Any subspace ~~submodule~~ of a Lie algebra closed under $[[xy]z]$ is a L.t.s.; the most important subspaces ~~submodules~~ of this type which are not subalgebras are the subspaces ~~modules~~ $\mathcal{L}_- = \{x,\ \alpha x = -x\}$ where $\alpha \in \mathrm{Aut}\,\mathcal{L}$, $\alpha^2 = \mathrm{id}$.

Exercise 1

Prove

3) If $\mathcal{T}$ together with $(x,y,z) \longmapsto \langle xyz \rangle$ is an associative triple system, then $\mathcal{T}$ together with

$$[xyz] := \langle xyz \rangle - \langle yxz \rangle - \langle zxy \rangle + \langle zyx \rangle$$

is a L.t.s. An important example of this type is F_n the vector space of column vectors over a field F; $\langle xyz \rangle = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} (y_1, \dots, y_n) \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} x_1y_1z_1 + x_1y_2z_2 + \cdots + x_1y_nz_n \\ \cdots \\ x_ny_1z_1 + x_ny_2z_2 + \cdots + x_ny_nz_n \end{pmatrix}$

Exercise 2

Prove

4) Let $\mathfrak{A}$ be a commutative algebra over Φ $= \mathbb{R}$ or $\mathbb{C}$ with multiplication $(x,y) \longmapsto xy = L(x)y$. Set $D(x,y) = [L(x),L(y)]$. Assume

(6.2) $$[D(x,y),\ D(u,v)] = D(D(x,y)u,v) + D(u,\ D(x,y)v) \quad \text{for all } x,y,u,v \in \mathfrak{A}.$$

If $\mathcal{T}$ is a subspace ~~submodule~~ of $\mathfrak{A}$ closed under $[xyz] = D(x,y)z$ then $\mathcal{T}$ together with $(x,y,z) \longmapsto [xyz]$ is a <u>Lie triple system.</u> The most important examples for this type of algebras are the Jordan algebras.

<u>Exercise</u>. Verify that the given examples really are L.t.s.'s.

(6.1 i) implies (replace x by x+y)

(6.2) $$[xyz] = - [yxz].$$

Define $L(x,y)$, $R(z,y)$, $P(x,z) \in \mathrm{End}_\Phi \mathfrak{T}$ (see Chapter III) by $[xyz] = L(x,y)z = R(z,y)x = P(x,z)y$. We see that (6.1) is equivalent to

(6.3)
$$\begin{aligned} &\text{(i)} && L(x,x) = 0 \qquad (\Rightarrow L(x,y) = - L(y,x)) \\ &\text{(ii)} && L(x,y) = R(x,y) - R(y,x) \\ &\text{(iii)} && [L(x,y), L(u,v)] = L([xyu],v) + L(u,[xyv]). \end{aligned}$$

U of F

<u>Lemma 1</u>. <u>A ~~submodule~~ subspace $\mathfrak{U}$ of $\mathfrak{T}$ is an ideal of $\mathfrak{T}$, iff</u> $[\mathfrak{U}\mathfrak{T}\mathfrak{T}] \subset \mathfrak{U}$.

<u>Proof</u>. Clearly the condition is necessary. Since $[\mathfrak{U}\mathfrak{T}\mathfrak{T}] \subset \mathfrak{U}$ imples $[\mathfrak{T}\mathfrak{U}\mathfrak{T}] \subset \mathfrak{U}$ (by (6.2)) and then $[\mathfrak{T}\mathfrak{T}\mathfrak{U}] \subset \mathfrak{U}$ by the Jacobi identity, we see that the given condition is also sufficient.

6.2. Let $\mathfrak{T}$ be a Lie triple system. We recall that $D \in \mathrm{End}_\Phi \mathfrak{T}$ is a derivation, if

(6.4) $$[D,L(x,y)] = L(Dx,y) + L(x,Dy).$$

(6.3iii) shows, that all $L(x,y)$, $x,y \in \mathfrak{T}$ are derivations. Let ~~$\mathfrak{L}$~~ $\mathcal{H}$ be the ~~submodule~~ subspace of ~~$\mathfrak{D}(\mathfrak{T})$~~ $\mathcal{D}(F)$ (derivation algebra of $\mathfrak{T}$) ~~generated~~ spanned by all $L(x,y)$, $x,y \in \mathfrak{T}$. Another interpretation of (6.4) gives:

<u>Lemma 2</u>. $\mathfrak{L}$ <u>is an ideal of</u> $\mathfrak{D}(\mathfrak{T})$.

$\mathcal{D}(F)$ is a Lie algebra $\quad [A,B] = AB - BA$

in particular $\mathcal{H}$ is a Lie subalgebra of $\mathcal{D}(F)$

G $\qquad$ D(F) $\qquad$ H

Let $\mathfrak{G}$ be a subalgebra of $\mathfrak{D}(\mathfrak{F})$ containing $\mathfrak{H}$. We consider

$$\mathfrak{L}(\mathfrak{G},\mathfrak{F}) = \mathfrak{G} \oplus \mathfrak{F}$$

and define for elements $X_i = H_i \oplus x_i$, $H_i \in \mathfrak{G}$, $x_i \in \mathfrak{F}$ $(i = 1,2)$ a product.

(6.5) $$[X_1, X_2] := \Big(\overbrace{[H_1, H_2] + L(x_1, x_2)}^{\in G}\Big) \oplus \Big(\overbrace{H_1 x_2 - H_2 x_1}^{\in F}\Big).$$

The following result is fundamental.

<u>Theorem 1</u>. If $\mathfrak{F}$ <u>is a Lie triple system</u>, $\mathfrak{G}$ <u>a subalgebra of</u> $\mathfrak{D}(\mathfrak{F})$ <u>containing</u> $\mathfrak{H}$, <u>then</u>

(i) $\mathfrak{L}(\mathfrak{G},\mathfrak{F}) = \mathfrak{G} \oplus \mathfrak{F}$ <u>together with the product</u> (6.5) <u>is a Lie algebra</u>,

(ii) $\Theta: H \oplus x \longmapsto (-H) \oplus x$ <u>defines an involution of</u> $\mathfrak{L}$,

(iii) $\mathfrak{L}(\mathfrak{H},\mathfrak{F}) = \mathfrak{H} \oplus \mathfrak{F}$ <u>is an ideal of</u> $\mathfrak{L}(\mathfrak{G},\mathfrak{F})$,

(iv) <u>if</u> $x,y,z \in \mathfrak{F}$, <u>then</u> $[xyz] = [[x,y],z]$,

(v) <u>if</u> $1/2 \in \Phi$ <u>then</u> $\mathfrak{F} = \{X \in \mathfrak{L}(\mathfrak{G},\mathfrak{F});\ \Theta X = X\}$.

<u>Proof</u>. Clearly $[X,X] = 0$ for all $X \in \mathfrak{L}$. We have to show $\mathfrak{J}(X_1,X_2,X_3) = [[X_1,X_2], X_3] + [[X_2,X_3], X_1] + [[X_3,X_1], X_2] = 0$ for all $X_i \in \mathfrak{L}$. It is sufficient to show this equality only for $(X_1,X_2,X_3) \in \mathfrak{U} \times \mathfrak{V} \times \mathfrak{W}$, $U \times V \times W$ where $\mathfrak{U}, \mathfrak{V}, \mathfrak{W}$ is either $\mathfrak{G}$ or $\mathfrak{F}$. Since $\mathfrak{G}$ is a subalgebra of $\mathfrak{D}(\mathfrak{F})$ we get $\mathfrak{J}(\mathfrak{G},\mathfrak{G},\mathfrak{G}) = 0$. If $H_i \in \mathfrak{G}$, $x \in \mathfrak{F}$ we get $[[H_1,H_2],x] = [H_1,H_2]x = H_1(H_2x) - H_2(H_1x) = [H_1,[H_2,x]] - [H_2[H_1,x]]$. This shows $\mathfrak{J}(\mathfrak{G},\mathfrak{G},\mathfrak{F}) = 0$, then by cyclic permutation $\mathfrak{J}(\mathfrak{G},\mathfrak{F},\mathfrak{G}) = \mathfrak{J}(\mathfrak{F},\mathfrak{G},\mathfrak{G}) = 0$. Using (6.5) and $\mathfrak{G} \subset \mathfrak{D}(\mathfrak{F})$ we get

$$[[H,x],y] + [[x,y],H] + [[y,H],x] = L(Hx,y) + [L(x,y),H] + L(x,Hy) = 0.$$

Hence $\mathfrak{J}(\mathfrak{g},\mathfrak{T},\mathfrak{T}) = 0$ and also $\mathfrak{J}(\mathfrak{T},\mathfrak{g},\mathfrak{T}) = \mathfrak{J}(\mathfrak{T},\mathfrak{T},\mathfrak{g})$ $= 0$. Finally $\mathfrak{J}(x,y,z) = [[x,y],z] + [[y,z],x] + [[z,x],y]$

$$= L(x,y)z + L(y,z)x + L(z,x)y = 0, \text{ by (6.1 ii).}$$

The other statements are easily verified (using definitions and lemma 2).

The Lie algebra $\mathfrak{L} = \mathfrak{L}(\mathfrak{H},\mathfrak{T}) = \mathfrak{H} \oplus \mathfrak{T}$ is called the <u>standard imbedding</u> of $\mathfrak{T}$, Θ is called the <u>main involution</u> of $\mathfrak{L}$.

<u>Examples</u>. 1) Let F be a field, $\mathfrak{T} = F_n$ the L.t.s. of column vectors over F (see ex. 3, p.43) We take as triple product $[xyz] = yx^t z - xy^t z$ and get $L(x,y)z = (yx^t - xy^t)z$. Consequently we can identify $L(x,y)$ with the $n \times n$ matrix $yx^t - xy^t$. The space spanned by these matrices is the space of all $n \times n$ skew symmetric matrices. We define a mapping of the standard imbedding $\mathfrak{H} \oplus \mathfrak{T}$ onto the Lie algebra of all $(n+1) \times (n+1)$ skew symmetric matrices by

$$A \oplus x \longmapsto \begin{pmatrix} A & x \\ -x^t & 0 \end{pmatrix}$$

Exercise 3

This is a (well defined) 1-1 linear map onto. It is an easy computation (and is left as an exercise) that the given map is a Lie algebra homomorphism, hence an isomorphism.

2) The above example may be generalized as follows. We define on $F^{(p,q)}$, the space of all $p \times q$ matrices the triple composition by (see ex. 3, p.43)

$$[ABC] = BA^tC - AB^tC - CB^tA + CA^tB.$$