V. Lie Algebras

the field, $\mathbb{R}$ or $\mathbb{C}$

5.1. We recall that an algebra $\mathcal{L}$ over Φ (with multiplication $(x,y) \mapsto [xy]$) is called a Lie algebra, if

(5.1) $$[xx] = 0$$

(5.2) $$[[xy]z] + [[yz]x] + [[zx]y] = 0 \quad \text{(Jacobi identity)}$$

for all $x,y,z \in \mathcal{L}$.

In Lie algebras (and only in Lie algebras) one denotes the left-multiplications by $\operatorname{ad}x$, $(\operatorname{ad}x)y = [xy]$. (5.1) implies

(5.3) $$[xy] = -[yx],$$

and the Jacobi identity then may be written as

$$\operatorname{ad}[xy] = [\operatorname{ad}x, \operatorname{ad}y].$$

Let $\mathcal{L}$ be a Lie algebra.

Lemma 1. *If $\mathfrak{A}, \mathfrak{B}$ are ideals of $\mathcal{L}$, then $[\mathfrak{A}\mathfrak{B}]$ is an ideal of $\mathcal{L}$.*

Proof. We need only prove $[a[uv]] \in [\mathfrak{A}\mathfrak{B}]$ for $a \in \mathcal{L}$, $u \in \mathfrak{A}$, $v \in \mathfrak{B}$. But this is immediately seen from the Jacobi identity.

Corollary. *If $\mathcal{L}$ $= \mathfrak{B}$! is an ideal of $\mathcal{L}$, then the "derived ~~modules~~ subspaces" $\mathcal{L}^{(0)} = \mathcal{L}$, $\mathcal{L}^{(n+1)} = [\mathcal{L}^{(n)}\mathcal{L}^{(n)}]$ and the powers of $\mathcal{L}^1 = \mathcal{L}$, $\mathcal{L}^{n+1} = [\mathcal{L}^n\mathcal{L}]$ are ideals of $\mathcal{L}$.* If $\mathcal{L}$ is Noetherian (i.e. has a.c.c. ascending chain condition on ideals) then there exists a unique maximal solvable ideal of $\mathcal{L}$. the radical of $\mathcal{L}$ (see 1.5). Also there exists a unique maximal nilpotent

see Meyberg pp. 5–6 for (solvable) radical
see Meyberg p. 7 for nilradical (nilpotent)

ideal in $\mathcal{L}$, the <u>nilradical</u>.

Exercise 1

Exercise. a) $[\mathcal{L}^m, \mathcal{L}^n] \subset \mathcal{L}^{m+n}$ (a) induction on m, Jacobi identity

b) $\mathcal{L}^{(n)} \subset \mathcal{L}^{2^n}$ (b) induction on n, Jacobi identity, & (a)

Exercise 2 c) The nilradical is contained in the radical.

We assume now, that $\mathcal{L}$ is a <u>finite dimensional</u> Lie algebra over a field F. In this case, there is a canonical bilinear form λ on $\mathcal{L}$, the so-called <u>Killing form</u>, defined by

$$\lambda(x,y) = \text{trace } (\text{ad}x)(\text{ad}y).$$

$\lambda(x,y) = \lambda(y,x)$ $\qquad$ $\lambda([x,y],z) = \lambda(x,[y,z])$

<u>Lemma 2</u>. (i) λ is <u>symmetric and associative</u>

(ii) $\lambda(\alpha x,y) = \lambda(x,\alpha^{-1}y)$ <u>for any</u> $\alpha \in$ <u>Aut</u> $\mathcal{L}$. $\quad$ α is linear and $\alpha([x,y]) = [\alpha(x),\alpha(y)]$

<u>Proof</u>. (i) the symmetry of λ is obvious. By definition and Jacobi identity $\lambda([xy],z) = \text{tr ad}[xy]\text{ad}z = \text{tr }(\text{ad}x\ \text{ad}y\ \text{ad}z - \text{ad}y\ \text{ad}x\ \text{ad}z)$

$$= \text{tr }(\text{ad}x\ [\text{ad}y, \text{ad}z]) = \text{tr ad}x\ \text{ad}[yz] = \lambda(x,[yz]).$$

(ii) $\alpha \in \text{Aut}\mathcal{L}$ is equivalent to $\alpha\ \text{ad}x\ \alpha^{-1} = \text{ad}(\alpha x)$. Therefore

$$\lambda(\alpha x,y) = \text{tr ad } \alpha x\ \text{ad}y = \text{tr } \alpha\ \text{ad}x\ \alpha^{-1}\ \text{ad}y$$

$$= \text{tr ad}x\ \alpha^{-1}(\text{ad}y)\alpha = \lambda(x, \alpha^{-1}y).$$

There is a fundamental result.

<u>Theorem 1</u>. (<u>CARTAN Criterion</u>). <u>Let</u> $\mathcal{L}$ <u>be a finite dimensional Lie algebra over a field of characteristic</u> 0. <u>Then</u> $\mathcal{L}$ <u>is semi-simple, iff the Killing form is non degenerate</u>.

i.e. the radical is 0.

<u>Exercise</u>: Read the proof of the Cartan Criterion in any book on Lie algebras.

An immediate application of theorem 1 and Dieudonné's theorem (1.9) is the following:

Theorem 2. If $\mathcal{L}$ is a finite dimensional semi-simple Lie algebra over a field of char. 0, then $\mathcal{L}$ is a direct sum of simple ideals.

Note: Condition (ii) in Dieudonné's theorem holds since $\mathcal{L}$ has no solvable ideal.

We shall give another application of the Cartan Criterion.

Theorem 3. (Zassenhaus). If $\mathcal{L}$ is as in theorem 2, then any derivation D of $\mathcal{L}$ is of the form $D = \mathrm{ad}\, d$, $d \in \mathcal{L}$.

Proof. Since the Killing form λ is non degenerate, there exists $d \in \mathcal{L}$ such that

$$\mathrm{trace}\left(D\ \mathrm{ad}x\right) = \lambda(d,x).$$

Let $E: = D - \mathrm{ad}\, d$, then E is a derivation and

(5.4) $$\mathrm{trace}\left(E\ \mathrm{ad}\ x\right) = \mathrm{trace}\left(D\ \mathrm{ad}\ x\right) - \mathrm{trace}\left(\mathrm{ad}\ d\ \mathrm{ad}\ x\right) = 0.$$

Then $\lambda(Ex,y) = \mathrm{tr}\left(\mathrm{ad}(Ex)\mathrm{ad}y\right) = \mathrm{tr}\left([E,\ \mathrm{ad}x]\mathrm{ad}y\right)$

$= \mathrm{tr}\left(E[\mathrm{ad}x,\ \mathrm{ad}y]\right) = \mathrm{tr}\left(E\ \mathrm{ad}[xy]\right) = 0$, by (5.4).

Since λ is non degenerate, we get $Ex = 0$ for all x or $D = \mathrm{ad}\, d$.