Complex Analysis Math 147—Winter 2006
Homework answers—Assignment 5;February 20, 2006

Part 1

1. $\mathbf{C}-\{5i,-5i\}$, $\mathbf{C}$, $\mathbf{C}-\{3+i,3-i\}$

   Each function is analytic in a (simply connected) domain containing C and the inside of C (for example $\{|z|<2.5\}$) so the integral is zero either by Cauchy's theorem. Alternatively, it is zero, since C is homotopic to a point in the domain.

2. First a correction: the integral is over the curve C.

   If D were simply connected, then C would be homotopic to a point in D and since $1/(z-z_0$ is analytic in D, the integral would be zero, a contradiction. Or simply by Cauchy's theorem since if D is simply connected, the integral would be zero.

3. $\int_C \frac{dz}{z^2+1} = \frac{i}{2}\int_C \frac{dz}{z+i} - \frac{i}{2}\int_C \frac{dz}{z-i}$, so $\int_{C_1} \frac{dz}{z^2+1} = 0 - 2\pi i\, i/2 = \pi$.
   If C_3 is the circle with center $-i$ and radius $1/2$ traversed in the counter clockwise direction once, then

   $\int_{C_3} \frac{dz}{z^2+1} = -\pi + 0 = -\pi$ so $\int_{C_2} \frac{dz}{z^2+1} = -\pi - (-\pi) = 0$.

4. $\int_C \frac{z\,dz}{(z+2)(z-1)} = \int_{|z-1|=1} \frac{(z/(z+2))\,dz}{z-1} + \int_{|z+2|=1} \frac{(z/(z-1))\,dz}{z+2} = 2\pi i(\frac{1}{3}) + 2\pi i(\frac{-2}{-3}) = 2\pi i.$
   Clockwise: $-2\pi i$

5. For $k \geq 2$, $1/z^k$ has an antiderivative on $\mathbf{C}-\{0\}$, so $\int_C \frac{dz}{z^k} = 0$. Since g is analytic in a domain containing C and its inside, by Cauchy's theorem, $\int_C g(z)\,dz = 0$. Thue $\int_C f(z)\,dz = A_1 \int_C \frac{dz}{z} = 2\pi i A_1$.

6. (a) Since $|z-1| \geq |z|-1$, if $|z| = R$, then $\left|\frac{1}{z^2(z-1)^3}\right| \leq \frac{1}{R^2(R-1)^3}$. Thus

   $|I(R)| \leq \frac{1}{R^2(R-1)^3}\, 2\pi R = \frac{2\pi}{R(R-1)^3}$.

   (b) $\int_C \frac{dz}{z^2(z-1)^3} = I(2) = I(R) \to 0$.

Part 2—Cain Chapter 6

1. If z_0 is inside C, then $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-z_0} = \frac{1}{2\pi i}\int_C \frac{g(z)\,dz}{z-z_0} = g(z_0)$.

2. (a) Let $f(s) = s^2+s+1$. Then $g(i) = 2\pi i f(i) = -2\pi$

   (b) $g(4i) = 0$ by Cauchy's theorem.

3. $\int_C \frac{e^{2z}}{(z+2)(z-2)}\,dz = 2\pi i\, \frac{e^{2z}}{z+2}\Big|_{z=2} = e^4\pi i/2$

4. $\int_\Gamma \frac{e^{2z}}{(z+2)(z-2)}\,dz = \int_{|z+2|=\epsilon} \frac{e^{2z}}{(z+2)(z-2)}\,dz + \int_{|z-2|=\epsilon} \frac{e^{2z}}{(z+2)(z-2)}\,dz = e^4\pi i/2 + 2\pi i\, \frac{e^{2z}}{z-2}\Big|_{z=-2} =$
   $(e^4-e^{-4})\pi i/2 = (\sinh 4)\pi i$.