

Complex Analysis Math 147—Winter 2006
Homework answers—Chapter 4; February 8, 2006

1. $1 + i/3$

2. $2\pi i$

3. Bye

4. $2 + 3i$

5. $7\pi/2$

(the actual value of the integral is $-17/3 + 5i$ which has modulus $\sqrt{514}/3 = 7.557$ and $7\pi/2$ is approximately 11).

6. $7/2 + 6i$.

7. $\int_C \cos(z/2) dz = F(\pi + 2i) - F(0) = 2 \cosh 1$, where $F(z) = 2 \sin(z/2)$.

8. (a) Let $F(z) = \log |z| + i \arg z$ where $0 < \arg z < 2\pi$. Thus F is defined on $D := \mathbf{C} - \{z = x + iy : x \geq 0, y = 0\}$. We need to prove that $F'(z)$ exists and equals $1/z$ for all $z \in D$. We shall use the following three facts to justify the steps following them.

(1) $z = \exp(F(z))$ for $z \in D$

[Proof: $\exp(\log |z| + i \arg z) = \exp(\log |z|) \exp(i \arg z) = |z| \exp(i \arg z) = z$]

(2) $F(z) \neq F(z_0)$ for $z, z_0 \in D$ and $z \neq z_0$

[Proof: If $F(z) = F(z_0)$, then $z = \exp(F(z)) = \exp(F(z_0)) = z_0$]

(3) F is continuous on D .

[Proof: It suffices to prove that $\arg z$ is continuous on D . The argument for this is similar to the solution of problem 12 on the review problems for chapters 1-3. First of all, the function $\arg z$ is not defined for $z = 0$. Let $z_0 = x_0$ be a positive real number. If $y > 0$, then $\arg(x_0 + iy) = \tan^{-1}(y/x_0) \rightarrow 0$ as $y \rightarrow 0$.

Also, if $y < 0$, then $\arg(x_0 + iy) = 2\pi - \tan^{-1}(-y/x_0) \rightarrow 2\pi$ as $y \rightarrow 0$. Therefore, $\lim_{z \rightarrow x_0} \arg z$ does not exist, and so $\arg z$ is not continuous at $z_0 = x_0$ if $x_0 > 0$.]

Now, using (1)–(3), and the fact that $\exp'(w) = \exp w$, we prove that $F'(z) = 1/z$:

$$\frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{\frac{z - z_0}{F(z) - F(z_0)}} = \frac{1}{\frac{\exp(F(z)) - \exp(F(z_0))}{F(z) - F(z_0)}}$$

so that

$$\lim_{z \rightarrow z_0} \frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{\lim_{F(z) \rightarrow F(z_0)} \frac{\exp(F(z)) - \exp(F(z_0))}{F(z) - F(z_0)}} = \frac{1}{\exp'(F(z_0))} = \frac{1}{\exp(F(z_0))} = \frac{1}{z_0}.$$

(b) Let $G(z) = \log |z| + i \arg z$ where $-\pi/4 < \arg z < 7\pi/4$. Thus G is defined on $E := \mathbf{C} - \{z = x + iy : x = -y \geq 0\}$. Prove that $G'(z)$ exists and equals $1/z$ for all $z \in E$.

Proof: We shall use the following three facts to justify the steps following them.

- (1) $z = \exp(G(z))$ for $z \in E$
- (2) $G(z) \neq G(z_0)$ for $z, z_0 \in E$ and $z \neq z_0$
- (3) G is continuous on E .

Now

$$\frac{G(z) - G(z_0)}{z - z_0} = \frac{1}{\frac{z - z_0}{G(z) - G(z_0)}} = \frac{1}{\frac{\exp(G(z)) - \exp(G(z_0))}{G(z) - G(z_0)}}$$

so that

$$\lim_{z \rightarrow z_0} \frac{G(z) - G(z_0)}{z - z_0} = \frac{1}{\lim_{G(z) \rightarrow G(z_0)} \frac{\exp(G(z)) - \exp(G(z_0))}{G(z) - G(z_0)}} = \frac{1}{\exp'(G(z_0))} = \frac{1}{\exp(G(z_0))} = \frac{1}{z_0}.$$

(c) $\int_{C_1} \frac{1}{z} dz = \text{Log}(i) - \text{Log}(-i) = \pi i$

(d) $\int_{C_2} \frac{1}{z} dz = F(i) - F(-i) = -\pi i$, where $F(z) = \log |z| + i \arg z$ and $0 < \arg z < 2\pi$.

9. The curve C is given by the function $\gamma(t) = \exp(-it)$, $0 \leq t \leq 2\pi$. Thus $\int_C \frac{1}{z} dz = \int_0^{2\pi} \frac{\gamma'(t) dt}{\gamma(t)} = \int_0^{2\pi} (-i) dt = -2\pi i$.

10. (a) $H'(z) = \exp(c \text{Log } z) c/z$. Since $H(z) = \exp(c \text{Log } z)$ and since $z^c z^d = z^{c+d}$ (proof?), this can be written as $H'(z) = cz^{c-1}$.

(b) $\{z \in \mathbf{C} : -\pi/4 < \arg z < \pi\}$

(c) $(1 + e^{-\pi})(1 - i)/2$

11. A polynomial has an antiderivative everywhere.