Complex Analysis Math 147—Winter 2006
Homework answers—Chapter 3;January 30, 2006

1. $\exp(z+2\pi i) = \exp z \exp 2\pi i$ and $\exp 2\pi i = e^0 \cos 2\pi + ie^0 \sin 2\pi = 1$

2. From $\exp(z+w) = \exp z \exp w$ you get $\exp z \exp(-z) = 1$ or $\exp(-z) = 1/\exp z$. Hence, $\exp z/\exp w = \exp z \exp(-w) = \exp(z-w)$.

3. $\exp z = e^x \cos y + ie^x \sin y$, so $|\exp z|^2 = (e^x \cos y)^2 + (e^x \sin y)^2 = (e^x)^2(\cos^2 x + \sin^2 x) = (e^x)^2$, and $|\exp z| = e^x$.

   Since $\arg w = \theta$ if $w = |w|(\cos\theta + i\sin\theta)$, y is an argument of $\exp(x+iy)$. But any two arguments differ by an integral multiple of 2π, so every argument of $\exp(x+iy)$ is of the form $y + 2k\pi$ for some $k \in \mathbf{Z}$.

4. $\exp(x+iy) = -1$ means $e^x \cos y = -1$ and $e^x \sin y = 0$. From the latter, $\sin y = 0$ so $y = k\pi$ with $k \in \mathbf{Z}$ Now $\cos k\pi = 1$ if k is even and $\cos k\pi = -1$ if k is odd. So putting $y = k\pi$ in $e^x \cos y = -1$ shows that $x = 0$ and $y = k\pi$ with k odd. So $\exp(z) = -1$ if and only if $z = ik\pi$ with k odd.

5. $\exp z = 1 + i$ is the same as $e^x \cos y = 1$ and $e^x \sin y = 1$, which is the same as $e^x = \sqrt{2}$ and $\tan y = 1$. Thus $x = \log\sqrt{2}$ and $y = \pi/4 + 2\pi k$ with $k \in \mathbf{Z}$. Hence $\exp z = 1 + i$ if and only if $z = \log\sqrt{2} + i(\pi/4 + 2\pi k)$ with $k \in \mathbf{Z}$.

6. $\exp z = e^x \cos y + ie^x \sin y$ is never zero since $\cos y$ and $\sin y$ are never both zero. On the other hand, if $w \neq 0$, then $\exp z = w$ if and only if $z = \log|w| + i\arg w$. Here $\arg w$ represents all the arguments of $w \neq 0$, for example $\arg w = \{\operatorname{Arg} w + 2\pi k : k \in \mathbf{Z}\}$.

7. bye

8. $\sin(z+2\pi) = \sin z \cos 2\pi + \cos z \sin 2\pi = \sin z$

   $\cos(z+2\pi) = \cos z \cos 2\pi - \sin z \sin 2\pi = \cos z$

   $\sin(z+\pi/2) = \sin z \cos \pi/2 + \cos x \sin \pi/2 = \cos z$

9. $\sin z = \sin x \cosh y + i \cos x \sinh y$, so $|\sin z|^2 = (\sin x \cosh y)^2 + (\cos x \sinh y)^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y = \sin^2 x(1 + \sinh^2 y) + \cos^2 x \sinh^2 y = \sin^2 x + (\sin^2 x + \cos^2 x)\sinh^2 y = \sin^2 x + \sinh^2 y$.

   Similarly, $|\cos z|^2 = \cos^2 x + \sinh^2 y$.

10. $\sin z = 0$ is the same as $\exp(2iz) = 1$, so by problem 6, $2iz = \log 1 + i \arg 1 = \{2k\pi i : k \in \mathbf{Z}\}$ so that $z = k\pi$ with $k \in \mathbf{Z}$.

11. $\cos z = \cos x \cosh y - i \sin x \sinh y$, so that $\cos z = 2$ if and only if $\cos x \cosh y = 2$ and $\sin x \sinh y = 0$. Looking at the last equation, either $\sinh y = 0$ or $\sin x = 0$. If $\sinh y = 0$, then $e^{2y} = 1$ so $y = 0$ and $\cos x = 2$ which is impossible. Therefore $\sin x = 0$ so that $x = k\pi$ for $k \in \mathbf{Z}$ and from $\cos x \cosh y = 2$ you get

$\cosh y = 2/\cos k\pi$, that is, $\cosh y = 2$ if k is even and $\cosh y = -2$ if k is odd. But $\cosh y$ is always positive, so $\cos z = 2$ if and only if $z = 2k\pi + i\cosh^{-1}(2)$, $k \in \mathbf{Z}$ (there are two values for $\cosh^{-1}(2)$).

12. YES.

    $\log i = \log|i| + i\arg i = \{i(\pi/2 + 2\pi k) : k \in \mathbf{Z}\}$ so

    $(1/2)\log i = \{i(\pi/4 + k\pi : k \in \mathbf{Z}\}$ and

    $i^{1/2} = \exp((1/2)\log i) = \{\exp((1/2)i(\pi/2 + 2\pi k) : k \in \mathbf{Z}\}$

    $= \{\exp(i(\pi/4 + k\pi)) : k \in \mathbf{Z}\}$ so that

    $\log(i^{1/2}) = \log|i^{1/2}| + i\arg(i^{1/2}) = \{i(\pi/4 + k\pi) : k \in \mathbf{Z}\}$.

13. NO.

    $\log(i^2) = \log(-1) = \log|-1| + i\arg(-1) = \{i(\pi + 2k\pi) : k \in \mathbf{Z}\}$

    From problem 12,

    $2\log i = \{2i(\pi/2 + 2\pi k) : k \in \mathbf{Z}\} = \{i(\pi + 4\pi k) : k \in \mathbf{Z}\}$, so $\log(i^2) \neq 2\log i$.

14. We begin by writing $\arg z = \{\operatorname{Arg} z + 2\pi k : k \in \mathbf{Z}\}$. Then

    $z^{1/2} = \exp((1/2)\log z) = \exp((1/2)(\log|z| + i\arg z))$

    $= \exp((1/2)(\log|z|)\exp(i(\arg z)/2) = \{|z|^{1/2}\exp(i((\operatorname{Arg} z)/2) + k\pi) : k \in \mathbf{Z}\}$, and

    $\log(z^{1/2}) = \log|z^{1/2}| + i\arg(z^{1/2}) = \{\log|z|^{1/2} + i((\operatorname{Arg} z)/2 + k\pi) : k \in \mathbf{Z})\}$.

15. It is useful to use the chain rule here. $g(z) = z^2 + 1$ is differentiable everywhere and $f(z) := \operatorname{Log} z$ is differentiable at every z in the set D which is the complement of the non-positive real axis, that is $D = \mathbf{C} - \{z = x + iy : y = 0 \text{ and } x \leq 0\}$ Hence $\operatorname{Log}(z^2+1) = f(g(z))$ is differentiable at every z for which $z^2 + 1 \in D$, that is, on the set $\mathbf{C} - \{z = x + iy : x = 0 \text{ and } |y| \geq 1\}$. Since this set is an open set, $\operatorname{Log}(z^2 + 1)$ is analytic there.

16. The principal value of i^i is $\exp(i\operatorname{Log} i) = \exp(i(i\operatorname{Arg} i)) = e^{-\pi/2}$.

    The principal value of $(1-i)^{4i}$ is

    $\exp(4i\operatorname{Log}(1-i)) = \exp(4i[\log|1-i| + i\operatorname{Arg}(1-i)])$

    $= \exp(4i(\log\sqrt{2} - i\pi/4)) = \exp(\pi + 2i\log 2)$.

17. $i^i = \exp(i\log i) = \exp(i(\log|i| + i\arg i)) = \exp(-\arg i)$

    $= \{\exp(-(\pi/2 + 2k\pi)) : k \in \mathbf{Z}\}$. These are positive numbers so

    $|i^i| = i^i = \{\exp(-(\pi/2 + 2k\pi)) : k \in \mathbf{Z}\}$.