

Complex Analysis Math 147—Winter 2006
Homework answers—Chapter 2; January 24, 2006

1. (a) The line segment from $4-6i$ to $7-5i$; the Cartesian equation is $x-4 = 3y+18$
 (b) The line segment from w to z ; if you insist, the equation would be

$$\frac{x - \Re w}{\Re z - \Re w} = \frac{y - \Im w}{\Im z - \Im w}$$

where $w = \Re w + i\Im w$, $z = \Re z + i\Im z$.

2. $\gamma(t) = t + i(4t^3 - 1)$, $0 \leq t \leq 10$.
3. $\gamma(t) = 2 \cos t + 3 + i2(\sin t - 1)$, $0 \leq t \leq 2\pi$.
4. bye
5. f is continuous, so $\lim_{z \rightarrow 3+2i} f(z) = f(3+2i) = 18 - i$
6. $\lim_{z \rightarrow z_0} (f(z) - f(z_0)) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \cdot \lim_{z \rightarrow z_0} (z - z_0) = f'(z_0) \cdot 0 = 0$.
7. $f(z) - f(z_0) = \bar{z} - \bar{z}_0 = (x - x_0) - i(y - y_0)$, where $z = x + iy$ and $z_0 = x_0 + iy_0$.
 First let z approach z_0 horizontally, so that $z = x + iy_0$. Then

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{x - x_0}{x - x_0} = 1$$

so that if $f'(z_0)$ existed, it would equal 1. On the other hand, if z approaches z_0 vertically, so that $z = x_0 + iy$, then

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{-i(y - y_0)}{i(y - y_0)} = -1$$

so that if $f'(z_0)$ existed, it would equal -1. Thus $f'(z_0)$ does not exist at any point z_0 .

8. This is a polynomial in z and so differentiable everywhere, and $f'(z) = 3(2 + i)z^2 - 2iz + 4$.
9. NO.

$$\frac{f(z) - f(0)}{z - 0} = \frac{(\bar{z})^2}{z^2} = \frac{(x - iy)^2}{(x + iy)^2} = \frac{x^2 - y^2 - 2ixy}{x^2 - y^2 + 2ixy}$$

Along the line $y = x$, this approaches -1 , and along the line $y = 0$, this approaches 1. So $f'(0)$ does not exist.

10. The Cauchy-Riemann equations are satisfied at $z = i$ and nowhere else. The partial derivatives are continuous everywhere, and in particular at $z = i$, so f is differentiable at $z = i$ only, and hence analytic nowhere.

11. YES, the Cauchy-Riemann equations are satisfied at $(0,0)$. However, with $f = u + iv$, u_x and v_y are not continuous at $(0,0)$. This explains why $f'(0)$ does not exist. Here are some details:

We have $u(x, y) = (x^3 - 3xy^2)/(x^2 + y^2)$ for $(x, y) \neq (0, 0)$, and $u(0, 0) = 0$, and $v(x, y) = (y^3 - 3x^2y)/(x^2 + y^2)$ for $(x, y) \neq (0, 0)$, and $v(0, 0) = 0$. So

$$u_x(0, 0) = \lim_{x \rightarrow 0} (u(x, 0) - u(0, 0))/x = \lim_{x \rightarrow 0} x/x = 1 \text{ and}$$

$v_y(0, 0) = \lim_{y \rightarrow 0} (v(0, y) - v(0, 0))/y = \lim_{y \rightarrow 0} y/y = 1$. This shows that one of the two Cauchy-Riemann equations is satisfied at $(0, 0)$.

However, for $(x, y) \neq (0, 0)$, $u_x(x, y) = (x^4 - 3y^4 + 6x^2y^2)/(x^2 + y^2)^2$. If $(x, y) \rightarrow (0, 0)$ along the line $y = x$ then $u_x(x, x) = 4x^4/4x^4 = 1$; if $(x, y) \rightarrow (0, 0)$ along the line $x = 0$ then $u_x(0, y) = -3y^4/y^4 = -3$. Therefore $\lim_{(x,y) \rightarrow (0,0)} u_x(x, y)$ does not exist, and so u_x is not continuous at $(0, 0)$.

12. NOWHERE.

13. If $f = u + iv$, then $f'(z) = u_x(x, y) + iv_x(x, y)$ and therefore $u_x = v_y = v_x = u_y = 0$ in D . By the mean value theorem, for any two points z_1, z_2 in D which can be connected by a straight line lying in D , we can find points (x_3, y_3) and (x_4, y_4) on that straight line such that

$$\begin{aligned} f(z_2) - f(z_1) &= u(x_2, y_2) - u(x_1, y_1) + i(v(x_2, y_2) - v(x_1, y_1)) \\ &= u_x(x_3, y_3)(x_2 - x_1) + u_y(x_3, y_3)(y_2 - y_1) \\ &\quad + i[v_x(x_4, y_4)(x_2 - x_1) + v_y(x_4, y_4)(y_2 - y_1)]. \end{aligned}$$

Therefore $f(z_1) = f(z_2)$. Since D is connected, any two points z, w in D can be connected by a finite sequence of line segments lying in D , and it follows that $f(z) = f(w)$ and f is a constant.

14. The function $f = u + iv$ is defined on $\mathbf{C} - \{0\}$. The Cauchy-Riemann equations are satisfied everywhere on $\mathbf{C} - \{0\}$ and the partial derivatives u_x and u_y are continuous on $\mathbf{C} - \{0\}$. Therefore f is differentiable at every point of $\mathbf{C} - \{0\}$, and therefore analytic on $\mathbf{C} - \{0\}$.

15. NO. Let $f(z) = 1$ if $|z| < 1$ and $f(z) = 1 + i$ on $\{z : |z - 3| < 1\}$. So f is defined and non constant on the open set $D = \{z : |z| < 1\} \cup \{z : |z - 3| < 1\}$, but $\Re f$ is constant on D .

If you add the condition that D is connected, then from $u_x = u_y = 0$ the Cauchy-Riemann equations tell you that $v_x = v_y = 0$ and by Problem 13, f is a constant.

16. NO. Let $f(z) = 1$ if $|z| < 1$ and $f(z) = i$ on $\{z : |z - 3| < 1\}$.

However, again, if D is connected then f is a constant.

PROOF: $u^2 + v^2 = c$ implies that

$$uu_x + vv_x = 0 = uu_y + vv_y$$

Using the Cauchy-Riemann equations, this becomes

$$uv_y - vu_y = 0 = vv_y + uu_y$$

or in matrix form

$$\begin{bmatrix} u & -v \\ v & u \end{bmatrix} \begin{bmatrix} v_y \\ u_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The determinant of this matrix is $u^2 + v^2 = c$, so if $c = 0$ then $u = v = 0$ so $f = 0$ is constant. If $c \neq 0$, then by linear algebra, $v_y = u_y = 0$ and we can use Problem 13 again.