Complex Analysis
Math 147—Winter 2006
Homework answers—Chapter 1

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1. $(4-7i)(-2+3i) = 13+26i$, $(1-i)^3 = -2-2i$, $\frac{5+2i}{1+i} = \frac{7}{2} - \frac{3}{2}i$, $\frac{1}{i} = -i$.

2. $z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

3. PROPOSITION: If z and w are complex numbers and $zw = 0$, then either $z = 0$ or $w = 0$.

   PROOF: Let $z = x + iy$, $w = u + iv$ and suppose $zw = 0$. Then $zw = (ux - vy) + i(uy + vx)$ so that

   $$ux - vy = 0 \text{ and } vx + uy = 0. \tag{1}$$

   Either $u \neq 0$ or $u = 0$. In the first case, from (1), $x = vy/u$ and $v(vy/u)+uy = 0$ so $y(v^2/u + u) = 0$ and either $y = 0$ or $v^2/u + u = 0$. If $v^2/u + u = 0$, then $v^2 + u^2 = 0$ so that $w = 0$, as required. If $y = 0$, then $x = vy/u = 0$ and $z = 0$, as required. Finally, if $u = 0$ and $v \neq 0$, then from $vy = 0$ and $vx = 0$ we get $x = y = 0$ and $z = 0$. □

4. (a) If $z = x + iy$ and $w = u + iv$, then

   $$\overline{zw} = \overline{xu - yv + i(xv + yu)} = xu - yv - i(xv + yu)$$

   and

   $$\overline{z}\ \overline{w} = (x - iy)(u - iv) = xu - yv - i(yu + xv).$$

   (b) If $z = x + iy$ and $w = u + iv \neq 0$, then

   $$\overline{z/w} = \overline{(z/w)(\overline{w}/\overline{w})} = \overline{z\overline{w}/w\overline{w}} = \overline{z\overline{w}/|w|^2} = \overline{z\overline{w}\frac{1}{|w|^2}} = \overline{z}\ \overline{\overline{w}}\ \overline{\left(\frac{1}{|w|^2}\right)} = \overline{z}\ w\frac{1}{|w|^2}$$

   and

   $$\overline{z}/\overline{w} = \overline{z}\ w/\overline{w}\ w = \overline{z}\ w/|w|^2.$$

   (c) By the triangle inequality, $|z| = |z - w + w| \leq |z - w| + |w|$, so $|z| - |w| \leq |z-w|$. This is true for all z, w so interchanging z and w gives $|w|-|z| \leq |w-z|$. Then $\mid |z| - |w| \mid = \max\{|w| - |z|, |z| - |w|\} \leq |z - w|$.

5. If $z = x + iy$ and $w = u + iv$, then

$$|zw|^2 = (xu - yv)^2 + (xv + yu)^2 = x^2u^2 + y^2u^2 + x^2v^2 + y^2v^2$$

and

$$|z|^2|w|^2 = (x^2 + y^2)(u^2 + v^2) = x^2u^2 + y^2u^2 + x^2v^2 + y^2v^2.$$

Also

$$|z/w| = |z\ \overline{w}/|w|^2| = |z\ \overline{w}\ \frac{1}{|w|^2}| = |z|\ |\overline{w}|\ |\frac{1}{|w|^2}| = |z|\ |w|/|w|^2 = |z|/|w|.$$

6. (a) Circle with center $2 - 3i$ and radius 2.
   (b) Closed disk with center $-2i$ and radius 1.
   (c) Vertical line passing through the point (4,0).
   (d) Line with equation $-8x + 2y - 5 = 0$
   (e) Ellipse with foci at (-1,0) and (1,0), and equation $x^2/4 + y^3/3 = 1$
   (f) The empty set.

7. (a) $i = e^{i\pi(1/2+2k)}$ $(k \in \mathbf{Z})$
   (b) $1 + i = \sqrt{2}e^{i\pi(1/4+2k)}$ $(k \in \mathbf{Z})$
   (c) $-2 = 2e^{i\pi(1+2k)}$ $(k \in \mathbf{Z})$
   (d) $-3i = 3e^{i\pi(3/2+2k)}$ $(k \in \mathbf{Z})$
   (e) $\sqrt{3} + 3i = 2\sqrt{3}e^{i\pi(1/3+2k)}$ $(k \in \mathbf{Z})$

8. (a) $2e^{i3\pi} = -2$
   (b) $e^{i100\pi} = 1$
   (c) $10e^{i\pi/6} = 5 + i5\sqrt{3}$
   (d) $\sqrt{2}e^{i5\pi/4} = -1 - i$

9. (a) $(1 + i)(1 + i\sqrt{3}) = e^{i7\pi/12}$
   (b) $\cos(7\pi/12) = (1 - \sqrt{3})/2\sqrt{2}$, $\sin(7\pi/12) = (1 + \sqrt{3})/2\sqrt{2}$

10. $(-1 + i)^{100} = -2^{50}$

11. $1,\ -1/2 + i\sqrt{3}/2\ ,\ \ -1/2 - i\sqrt{3}/2$

12. $\sqrt{2 + \sqrt{2}} + i\sqrt{2 - 2\sqrt{2}}\quad ,\quad -\sqrt{2 + \sqrt{2}} + i\sqrt{2 - 2\sqrt{2}}\ ,$

    $-\sqrt{2 + \sqrt{2}} - i\sqrt{2 - 2\sqrt{2}}\quad ,\quad \sqrt{2 + \sqrt{2}} - i\sqrt{2 - 2\sqrt{2}}$