Complex Analysis Math 147—Winter 2008 Homework answers—Assignment 9,10; March 17, 2008

Assignment 9 For a fixed complex number a with $|a| < 1$, define a function φ_a by

$$\varphi_a(z) = \frac{z + a}{1 + \overline{a}z}.$$

Although $\varphi_a(z)$ is defined for all $z \neq -1/\overline{a}$, we shall consider it as a function on the closed unit disk $|z| \leq 1$. Prove the following statements.

(a) If $|z| < 1$ then $|\varphi_a(z)| < 1$.

Proof:

$$\begin{aligned} \left|\frac{z + a}{1 + \overline{a}z}\right| < 1 \quad &\Leftrightarrow \quad |z + a|^2 < |1 + \overline{a}z|^2 \\ &\Leftrightarrow \quad |z|^2 + |a|^2 < 1 + |az|^2 \\ &\Leftrightarrow \quad |z|^2(1 - |a|^2) < 1 - |a|^2 \\ &\Leftrightarrow \quad |z|^2 < 1 \\ &\Leftrightarrow \quad |z| < 1 \end{aligned}$$

(b) If $|z| = 1$ then $|\varphi_a(z)| = 1$.

Proof: Same steps as in (a).

(c) φ_a is a one to one function, that is, if $|z_1| < 1, |z_2| < 1$ and if $\varphi_a(z_1) = \varphi_a(z_2)$, then $z_1 = z_2$.

Proof:

$$\begin{aligned} \frac{z_1 + a}{1 + \overline{a}z_1} = \frac{z_2 + a}{1 + \overline{a}z_2} \quad &\Leftrightarrow \quad z_1 + |a|^2 z_2 = z_2 + |a|^2 z_1 \\ &\Leftrightarrow \quad z_1(1 - |a|^2) = z_2(1 - |a|^2) \\ &\Leftrightarrow \quad z_1 = z_2 \end{aligned}$$

(d) φ_a is an onto function, that is, if $|w_0| < 1$, then there is a z_0 with $|z_0| < 1$ and $\varphi_a(z_0) = w_0$.

Proof:

$$\begin{aligned} \frac{z_0 + a}{1 + \overline{a}z_0} = w_0 \quad &\Leftrightarrow \quad z_0 + a = w_0 + w_0\overline{a}z_0 \\ &\Leftrightarrow \quad z_0(1 - \overline{a}w_0) = w_0 - a \\ &\Leftrightarrow \quad z_0 = \frac{w_0 - a}{1 - \overline{a}w_0} \end{aligned}$$

(e) What is the inverse of φ_a?

Proof: From (d) $(\varphi_a)^{-1} = \varphi_{-a}$.

Assignment 10 Let f be an arbitrary analytic function on the unit disk $|z| < 1$ which is one to one and onto, that is, if $|z_1| < 1, |z_2| < 1$ and if $f(z_1) = f(z_2)$, then $z_1 = z_2$; and if $|w_0| < 1$, then there is a z_0 with $|z_0| < 1$ and $f(z_0) = w_0$. Prove the following statements.

(a) If $f(0) = 0$, then $f(z) = e^{i\theta} z$ for some real θ.

Proof: By Schwarz's lemma, $|f(z)| \leq |z|$ for all $|z| < 1$. The inverse of f also satisfies the hypothesis of Schwarz's lemma. Hence $|f^{-1}(z)| \leq |z|$ for all $|z| < 1$. Thus $|z| = |f^{-1}(f(z))| \leq |f(z)| \leq |z|$ for every $|z| < 1$. Thus $|f(z)| = |z|$ for some $z \neq 0$. By Schwarz's lemma again, $f(z) = cz$ for all $|z| < 1$ and some constant c with $|c| = 1$.

(b) If $f(0) = a \neq 0$, let $g(z)$ be defined by $g(z) = \varphi_{-a}(f(z))$. Then $g(z) = e^{i\theta} z$ for some real θ.

Proof: The function g is one-to-one and onto, analytic and $g(0) = 0$. So by (a), $g(z) = e^{i\theta} z$ for some real θ.

(c) The function f has the form

$$f(z) = e^{i\theta} \varphi_a(z),$$

for some θ real and $|a| < 1$.

Proof:

$$f(z) = \varphi_a(\varphi_{-a}(f(z)) = \varphi_a(g(z)) = \varphi_a(e^{i\theta} z) = \frac{e^{i\theta} z - a}{1 - \overline{a} e^{i\theta} z} = e^{i\theta} \frac{z - e^{-i\theta} a}{1 - \overline{a e^{-i\theta}} z}.$$

Thus $f(z) = e^{i\theta} \varphi_b(z)$, where $b = e^{-i\theta} a$.