Complex Analysis Math 147—Winter 2008
Homework answers—Assignment 8,11; March 17, 2008

Assignment 8 Use Taylor's theorem to prove the following "leap of faith" in the proof of Schwarz's lemma. If f is analytic on the open unit disk and $f(0) = 0$ and $|f(z)| \le 1$ for all $|z| < 1$, then the function g defined on the unit disk by $g(z) = f(z)/z$ for $z \ne 0$ and $g(0) = f'(0)$, is analytic at 0.

Proof: Note first that g is continuous on $|z| < 1$, since it is analytic for $z \ne 0$ and since $g(0) = f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{z\to 0} g(z)$.

Let $f(z) = \sum_0^\infty a_n z^n$ be the Taylor series of f on $|z| < 1$. Since $a_0 = f(0) = 0$, $f(z) = zh(z)$ where $h(z) = a_1 + a_2 z + a_3 z^2 + \cdots$ is analytic on $|z| < 1$. Thus for $0 < |z| < 1$, $g(z) = f(z)/z = h(z)$. Thus g and h are continuous functions on $|z| < 1$ which agree on $0 < |z| < 1$. Then $g(0) = \lim_{z\to 0} g(z) = \lim_{z\to 0} h(z) = h(0)$. Thus $g = h$ is analytic on $|z| < 1$. □

Assignment 11 Use the identity theorem to prove the following statements

1. Find all entire functions f that satisfy $f(z) = e^x$ for $z = x \in \mathbf{R}$.

   Proof: If f is entire and $f(z) = e^x$ for $z = x \in \mathbf{R}$, then $g(z) := f(z) - \exp z$ is entire and vanishes on the real axis. By the identity theorem, $g(z) = 0$ for all $z \in \mathbf{C}$. Thus $\exp z$ is the only entire function with the property stated. □

2. Let f and g be analytic functions defined on a domain D and suppose $f(z)g(z) = 0$ for every $z \in D$. Show that either $f \equiv 0$ or $g \equiv 0$.

   Proof: Suppose that one of f and g is not identically zero, say $f(z_0) \ne 0$ for some $z_0 \in D$. Since f is continuous, there exists $R > 0$ such that $B(z_0, R) \subset D$ and $f(z) \ne 0$ for all $z \in B(z_0, R)$. Since $fg = 0$ throughout D, we must have $g(z) = 0$ for all $z \in B(z_0, R)$. Then by the identity theorem, g must be identically zero in D. □

3. Suppose that f is analytic on $\{|z| < 2\}$. Show that there must exist some positive integer n such that $f(1/n) \ne 1/(n + 1)$.

   Proof: Suppose that $f(1/n) = 1/(n + 1)$ for all $n \ge 1$. The function $g(z) := z/(1 + z)$, which is analytic on $|z| < 1$, satisfies $g(1/n) = 1/(n + 1)$ also. Thus $f - g$ vanishes on a set with a limit point in $|z| < 1$, so by the identity theorem, $f(z) = z/(z + 1)$ for all $|z| < 1$. But f is given to be analytic on $|z| < 2$, in particular at $z = -1$. This is a contradiction since $\lim_{z\to -1}(z/z + 1)$ doesn't exist. □