

Complex Analysis Math 147—Winter 2008
Solution to Assignment 7—March 17, 2008

Assignment 7 Complete the proof of Proposition 22.3 as follows. Suppose that $G = A \cup B$ where A, B are open, non-empty, and disjoint. Let z_0, z_1, \dots, z_n be a finite sequence of points in D with $z_0 = a, z_n = b$ and such that the line segments $[z_{k-1}, z_k] := \{sz_k + (1-s)z_{k-1} : s \in [0, 1]\}$ all lie in D . Choose one of these segments which has one endpoint in A and the other in B , and denote it by $[p, q]$. Then $[0, 1] = S \cup T$ where $S = \{s \in [0, 1] : sq + (1-s)p \in A\}$ and $T = \{t \in [0, 1] : tq + (1-t)p \in B\}$. S and T are each non-empty, since $0 \in S$ and $1 \in T$.

- (a) Prove that S is equal to a set of the form $\{0\} \cup S'$ where S' is open. Prove that T is equal to a set of the form $\{1\} \cup T'$ where T' is open.

Proof: Let $s_0 \in S - \{0\}$ so that $p_0 := s_0q + (1-s_0)p \in A$. Since A is open in \mathbf{C} , there exists $\delta > 0$ with $B(p_0, \delta) \subset A$. I claim that

$$(s_0 - \delta/2M, s_0 + \delta/2M) \subset S \text{ where } M = \max_j |z_j|.$$

Assuming this claim to be true, we have that $S - \{0\}$ is a union of open intervals, completing the proof of (a).

Proof of claim: If $s \in (s_0 - \delta/2M, s_0 + \delta/2M)$, then $|s - s_0| < \delta/2M$ and

$$|sq + (1-s)p - (s_0q + (1-s_0)p)| \leq |(s-s_0)q + (s_0-s)p| \leq 2|s-s_0|M < \delta.$$

Therefore $sq + (1-s)p \in B(p_0, \delta) \subset A$ and $s \in S$.

(The proof for T is similar)

- (b) Complete the proof of Proposition 22.3 by deriving a contradiction (Hint: Consider $\alpha = \sup S$ and the three cases $\alpha = 0, \alpha = 1, 0 < \alpha < 1$).

Proof: If $\alpha = 0$, then $S = \{0\}$ and $T = (0, 1]$. This cannot happen, since A is open; the argument is the same as the one on (a) which shows that $[0, \delta/2M) \subset S$.

If $\alpha = 1$, then $S = [0, 1)$ and $T = \{1\}$. This cannot happen, since B is open; the argument is the same as the one on (a) and shows that $(0, 1 - \delta/2M] \subset T$.

It must be that $0 < \alpha < 1$. If $\alpha \in S$, then $T = (\alpha, 1]$ and $S = [0, \alpha]$, which contradicts (a). If $\alpha \in T$, then $T = [\alpha, 1]$, which contradicts (a).

We have a contradiction in each of the three possibilities for α , so Proposition 22.3 is proved.