Complex Analysis Math 147—Winter 2006
Homework answers—Assignment 6;February 27, 2006

5. $\int_C \frac{\sin z}{z^2}\, dz = 2\pi i(\sin z)'(0) = 2\pi i$

6. (a) $\int_C \frac{1}{z^2+4}\, dz = \int_C \frac{1}{(z+2i)(z-2i)}\, dz = 2\pi i \left(\frac{1}{z+2i}\right)|_{z=2i} = \pi/2$

(b) $\int_C \frac{1}{(z^2+4)^2}\, dz = 2\pi i \left(\frac{1}{(z+2i)^2}\right)' (2i) = \pi/16$

7. Both are zero if w is outside C. If w is inside C, then

$\int_C \frac{f'(z)}{z-w}\, dz = 2\pi i f'(z) = 2\pi i \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-w)^2}\, dz$

8. (a) $\int_C \frac{e^{\alpha z}}{z}\, dz = 2\pi i \left(e^{\alpha z}\right)_{z=0} = 2\pi i$

(b) $2\pi i = \int_C \frac{e^{\alpha z}}{z}\, dz = \int_{-\pi}^{\pi} e^{\alpha e^{it}} i\, dt$ so

$2\pi = \int_{-\pi}^{\pi} e^{\alpha(\cos t + i \sin t)}\, dt = \int_{-\pi}^{\pi} e^{\alpha \cos t} e^{i\alpha \sin t}\, dt = \int_{-\pi}^{\pi} e^{\alpha \cos t} \left(\cos(\alpha \sin t) + i \sin(\alpha \sin t)\right)\, dt$

$= \int_{-\pi}^{\pi} e^{\alpha \cos t} \left(\cos(\alpha \sin t)\right)\, dt$ since 2π is real.

9. $g(z) := \exp(f(z))$ is an entire function, and $|g(z)| = \exp(\operatorname{Re} f(z)) \leq M$, so by Liouville's theorem, g is a constant c. Differentiate the equation $\exp(f(z)) = c$ to get $\exp(f(z))\, f'(z) = 0$ for every $z \in \mathbf{C}$. Since the exponential function never vanishes, $f'(z) = 0$ for every $z \in \mathbf{C}$ and since $\mathbf{C}$ is connected, f is a constant.

10. If $|w| > 3$ and $5w^4 + w^3 + w^2 - 7w + 14 = 0$, then

$1/|w| < 1/3$ and $5 + 1/w + 1/w^2 - 7/w^3 + 14/w^4 = 0$

so $5 = |1/w + 1/w^2 - 7/w^3 + 14/w^4|$ is at most $1/3 + 1/9 + 7/27 + 14/81 < 4$,

a contradiction, so $|w|$ is at most 3.

14. Since f is analytic and never zero, $g(z) := 1/f(z)$ is also analytic, so does not have a maximum, that is, there is no point z_0 such that $|1/f(z))| \leq |1/f(z_0)|$ for all z. This is the same as $|f(z)| \geq |f(z_0)|$ for all z so f does not have a minimum.

15. $g(z) := \exp(f(z))$ is analytic, and $|g(z)| = \exp(u(x,y)$, so if $u(x_0,y_0) \geq u(x,y)$ for all $z = x + iy$, then, with $z_0 = x_0 + iy_0$, $|\exp(f(z_0))| = \exp u(x_0,y_0) \geq \exp u(x,y) = |\exp(f(z))|$. By the maximum modulus theorem, $\exp f(z)$ is a constant, and therefore, since the domain is connected, f is a constant. Hence $u = \operatorname{Re} f$ is a constant.