get $(\rho 1 + v)y = \rho y \in \mathfrak{U} \cap V$ and we are back in the case, from which we derived $\mathfrak{U} = \mathfrak{A}$. We proved:

<u>If</u> dim V ≥ 2 <u>and</u> $q(x,y)$ is <u>a</u> <u>non</u> <u>degenerate</u> <u>bilinear</u> <u>form</u> <u>then</u> <u>the</u> <u>Jordan</u> <u>algebra</u> $\mathfrak{A} = F1 \oplus V$ <u>is</u> <u>simple</u>.

Next we show that the Jordan algebra $F1 \oplus V$, we considered above, is special. For this purpose we have to introduce the Clifford algebra $\mathcal{C}(V,q)$.

Let $\mathcal{T}(V)$ be the tensor algebra over V, that is

$\mathcal{T}(V) = \bigoplus_{i \geq 0} V^i$, where $V^0 := F1$ and $V^i = \bigotimes_{j=1}^{i} V$, the multiplication in $\mathcal{T}(V)$ is defined for the generators $\alpha \cdot x = \alpha x$,

$(a_1 \otimes \ldots \otimes a_s) \cdot (a_{s+1} \otimes \ldots \otimes a_r) = a_1 \otimes \ldots \otimes a_{s+1} \otimes . \otimes a_r$. (then linearly extended). It is obvious that $\mathcal{T}(V)$ is an associative algebra with unit element 1. Let $\mathfrak{K}$ be the ideal generated by $\{x \otimes x - q(x)1; x \in V\}$. The quotient algebra

$$\mathcal{C}(V,q) = \mathcal{T}(V)/\mathfrak{K}$$

is called the Clifford algebra of q.

Let $\pi : F1 \oplus V \to \mathcal{C}$ the canonical map $x \mapsto x + \mathfrak{K}$, then by the definition we have

$$\pi(\alpha 1 + x)^2 = (\alpha 1 + x) \cdot (\alpha 1 + x) + \mathfrak{K} = \alpha^2 1 + 2\alpha x + x \otimes x + \mathfrak{K}$$
$$= [\alpha^2 + q(x,x)]\, 1 + 2\alpha x + \mathfrak{K} = \pi((\alpha 1 + x)^2)$$

which implies, that $\pi(F1 \oplus V)$ is a (Jordan) subalgebra of $\mathcal{C}^+$, and $\pi : F1 \oplus V \to \mathcal{C}^+$ a homomorphism. One can show that π (restricted to $F1 \oplus V$) is 1 - 1. This shows that $F1 \oplus V$ is isomorphic to the subalgebra $\pi(F1 \oplus V)$ in $\mathcal{C}^+$, hence it is special.

8.3. Let $\mathfrak{d}$ be an arbitrary algebra over F with involution

$j : x \to \bar{x}$. By $\mathcal{D}_n$ we denote the algebra of $n\times n$ matrices with entries in $\mathcal{D}$. In $\mathcal{D}_n$ we have the standard involution $X \to \bar{X}^t$, where $\bar{X} = (\bar{\alpha}_{ij})$ if $X = (\alpha_{ij})$ and Y^t is the transposed of $Y \in \mathcal{D}_n$. (Verify that $X \to \bar{X}^t$ is an involution.) The space of symmetric elements relative to this involution is denoted by $\mathcal{H}(\mathcal{D}_n)$.

$\mathcal{H}(\mathcal{D}_n) = \{X \in \mathcal{D}_n; \quad X = \bar{X}^t\}$.

Clearly $X^oY = \frac{1}{2}(XY + YX) \in \mathcal{H}(\mathcal{D}_n)$ if $X,Y \in \mathcal{H}(\mathcal{D}_n)$ (XY denotes the usual matrix product). This shows, that $\mathcal{H}(\mathcal{D}_n)$ together with $(X,Y) \mapsto X^oY$ is an algebra. Without proof we state the following important result (see N. Jacobson, Structure and Representations of Jordan Algebras). Theorem 1 on p. 127

Theorem 2. For $n \geq 3$ $(\mathcal{H}(\mathcal{D}_n), o)$ is a Jordan algebra, iff either $\mathcal{D}$ is associative or $n = 3$ and $\mathcal{D}$ is alternative and any j-symmetric element α in $\mathcal{D}$, satisfies $(\alpha x)y = \alpha(xy)$ for all $x,y \in \mathcal{D}$.

DEF: An algebra $\mathcal{D}$ is called alternative, if

(8.1) $\quad x(xy) = x^2y$ and $(yx)x = yx^2$ for all $x,y \in \mathcal{D}$.

If $(\mathcal{D},j)$ is a simple pair and $\mathcal{D}$ an associative Artinian algebra, then $\mathcal{H}(\mathcal{D}_n)$ is a simple Jordan algebra. No reference given

8.4. In order to present a class of exceptional Jordan algebras we first have to introduce Cayley algebras.

Let $\mathcal{L}$ be an alternative algebra with unit element e and non degenerate quadratic form q such that

$$x^2 - t(x)x + q(x)e = 0$$

for all $x \in \mathcal{L}$, where $t(x) := q(x,e) = q(x + e) - q(x) - q(e)$.

For example F, or F$\oplus$F, or the algebra of 2×2 matrices over F have these properties, relative to $q(\alpha) = \alpha^2, q(\alpha \oplus \beta) = \alpha\beta$ or $q(a) = \det a$.

It is fairly easy to show that

$$x \to \bar{x} := t(x)e - x$$

defines an involution on $\mathcal{L}$. (Compare the following with the construction of the complex numbers from the reals.) Let $\mathcal{L}$ be as described above and $\mathcal{L}1$ an isomorphic copy of $\mathcal{L}$ (identify $e1$ with 1) and $\mu \in F, \mu \neq 0$. In the direct sum

$$(\mathcal{L}, \mu) = \mathcal{L} \oplus \mathcal{L}1$$

we define a product by

$$(x + y1)(u + v1) := (xu + \mu \bar{v}y) + (vx + y\bar{u})1.$$

A simple verification shows

$$(x + y1)^2 - t(x + y1)(x + y1) + q(x + y1)e = 0,$$

where $q(x + y1) := q(x) - \mu q(y)$, which is again non degenerate $(\mu \neq 0)$. But it is not clear whether the alternative laws (8.1) hold in $(\mathcal{L}, \mu)$. This is settled by the following result:

a) $(\mathcal{L}, \mu)$ <u>alternative</u>, <u>iff</u> $\mathcal{L}$ <u>associative</u>,

b) $(\mathcal{L}, \mu)$ <u>associative</u>, <u>iff</u> $\mathcal{L}$ <u>associative</u> <u>and</u> <u>commutative</u>,

c) $(\mathcal{L}, \mu)$ <u>commutative</u>, <u>iff</u> $\mathcal{L} = Fe$.

Therefore we can easily construct four classes of alternative algebras with the required properties. Starting with

$\mathcal{L}_0 = Fe$, and $\mu_1 \neq 0$ we get

$\mathcal{L}_1 = (Fe, \mu_1)$, which is commutative; then for $\mu_2 \neq 0$

$\mathcal{L}_2 = (Fe, \mu_1, \mu_2)$ is associative, and

$\mathcal{L}_3 = (Fe, \mu_1, \mu_2, \mu_3)$ is alternative $(\mu_3 \neq 0)$.

It can be shown that $\mathcal{L}_3$ is not associative, therefore an algebra $\mathcal{L}_4 = (\mathcal{L}_3, \mu)$, $\mu \neq 0$, would no longer be alternative. The indicated construction is called the Cayley-Dickson construction.

$\mathcal{L}_1$ is either a quadratic extension of Fe, or $\mathcal{L}_1 = Fe \oplus Fe$. $\mathcal{L}_2 = (Fe, \mu_1, \mu_2)$ is called a (generalized) quaternion algebra and $\mathcal{L}_3 = (Fe, \mu_1, \mu_2, \mu_3)$ is called a Cayley algebra (or octonion algebra).

Exercise: Choose an appropriate basis in $\mathcal{L}_i$ $(i = 1,2,3)$ and determine the multiplication table of this basis. (For more information about these algebras (and, of course, many other topics) see: Braun-Koecher, Jordan-Algebren; N. Jacobson, Structure and Representations of Jordan Algebras; and R.D. Schafer, An Introduction to Nonassociative Algebras.).

3 BOOKS

8.5. Now let $\mathcal{L}$ be a Cayley algebra, then $\mathcal{L}$ has an involution $x \to \bar{x} = t(x)e - x$, the symmetric elements then are obviously exactly the elements in Fe. But for $\alpha \in F$ we have trivially $\alpha(xy) = x(\alpha y)$. Therefore theorem 2 applies to show, that

$$\mathcal{H}(\mathcal{L}_3) = \left\{ X = \begin{pmatrix} \alpha_1 & x_1 & x_2 \\ \bar{x}_1 & \alpha_2 & x_3 \\ \bar{x}_2 & \bar{x}_3 & \alpha_3 \end{pmatrix} ; \ \alpha_i \in F,\ x_i \in \mathcal{L} \right\}$$

together with $X \circ Y = \frac{1}{2}(XY + YX)$ is a Jordan algebra. This algebra is simple and exceptional.

IX. Quadratic Jordan Algebras.

9.1. Let Φ be a commutative ring with unit element 1. A map of