by K. McCrimmon. Before presenting some fundamentals of his theory we shall study some examples of linear Jordan algebras.

VIII. Examples of Linear Jordan Algebras.

Throughout this chapter we assume $\frac{1}{2} \in \Phi$.

8.1. We already know, that for an associative algebra $\mathfrak{A}$ with multiplication $(x,y) \to xy$, the algebra $\mathfrak{A}^+$, i.e., $\mathfrak{A}$ together with $x \circ y := \frac{1}{2}(xy + yx)$, is a Jordan algebra. But then any vector subspace ~~submodule~~ $\mathcal{L}$ of $\mathfrak{A}$, closed under $(x,y) \mapsto x \circ y$, is also a Jordan algebra. A (linear) Jordan algebra is called <u>special</u>, if it is isomorphic to a (Jordan -) subalgebra of some $\mathfrak{A}^+$, $\mathfrak{A}$ associative. Which makes the theory more complicated, but more interesting, is the fact, that there are Jordan algebras which are not special. These are called <u>exceptional</u> Jordan algebras.

For the most interesting applications of Jordan algebras one needs simple algebras. Therefore we shall look for conditions on $\mathfrak{A}$ which force $\mathfrak{A}^+$ to be simple. Obviously any associative ideal of $\mathfrak{A}$ is an ideal of $\mathfrak{A}^+$. We shall show the converse. We start with:

<u>Lemma 1.</u> <u>If $\mathcal{L}$ is an ideal in $\mathfrak{A}^+$, then for all</u> $a,b \in \mathcal{L}$ <u>and</u> $x \in \mathfrak{A}$, $(ab + ba)x - x(ab + ba) \in \mathcal{L}$.

<u>Proof.</u> An immediate verification shows

$x(ab + ba) - (ab + ba)x = a(xb - bx) + (xb - bx)a + (xa - ax)b + b(xa - ax)$.

Since $a,b \in \mathcal{L}$, we have that $ya + ay$ and $yb + by$ are elements in $\mathcal{L}$ for all $y \in \mathfrak{A}$; so the right hand side of the above equation is in $\mathcal{L}$, for all $x \in \mathfrak{A}$. This already proves the lemma. An element

$x \in \mathfrak{A}$ is called trivial, if $x\mathfrak{A}x = 0$.

<u>Theorem 1.</u> <u>If $\mathfrak{A}$ has no trivial elements</u> $\neq 0$, <u>then any non-zero ideal $\mathfrak{B}$ of $\mathfrak{A}^+$ contains a non-zero ideal of $\mathfrak{A}$.</u>

<u>Proof.</u> Let $\mathfrak{B} \neq 0$ be an ideal of $\mathfrak{A}^+$. By lemma 1 we get for any $x \in \mathfrak{A}$, $xc - cx \in \mathfrak{B}$, where $c = ab + ba$, $a,b \in \mathfrak{B}$. Since $c \in \mathfrak{B}$, we have $xc + cx \in \mathfrak{B}$, consequently $xc \in \mathfrak{B}$ ($\frac{1}{2} \in \Phi$!) for all $x \in \mathfrak{A}$. But then again $(xc)y + y(xc) \in \mathfrak{B}$ for all y and therefore $xcy \in \mathfrak{B}$ for all $x,y \in \mathfrak{A}$ since we already showed $y(xc) = (yx)c \in \mathfrak{B}$. Then we have $\mathfrak{A}c\mathfrak{A} \subset \mathfrak{B}$. Since $\mathfrak{A}c\mathfrak{A}$ is an ideal in $\mathfrak{A}$, we are done, unless $\mathfrak{A}c\mathfrak{A} = 0$. In this case $c\mathfrak{A}c\mathfrak{A}c\mathfrak{A}c = 0$, which forces $c\mathfrak{A}c = 0$ and then $c = 0$, since $\mathfrak{A}$ has no trivial elements. If we can show, that for some $a,b \in \mathfrak{B}$ the element $c: = ab + ba \neq 0$, then by the foregoing $\mathfrak{A}c\mathfrak{A} \neq 0$. Therefore assume $ab + ba = 0$ for all $a,b \in \mathfrak{B}$. Then in particular $a^2 = 0$ and $2\,axa = a(ax + xa) + (ax + xa)a = 0$ since $ax + xa \in \mathfrak{B}$. This shows $a\mathfrak{A}a = 0$. Again our assumption implies $a = 0$, which contradicts $\mathfrak{B} \neq 0$.

<u>Corollary:</u> <u>If $\mathfrak{A}$ is a simple associative algebra then $\mathfrak{A}^+$ is a simple Jordan algebra.</u>

<u>Proof.</u> Firstly we note that $x\mathfrak{A} = 0$ implies that $\mathfrak{A}x$ is an ideal of $\mathfrak{A}$. Since $\mathfrak{A}x = \mathfrak{A}$ would imply $\mathfrak{A}^2 = \mathfrak{A}x\mathfrak{A} = 0$ we have $\mathfrak{A}x = 0$. Then Φx is an ideal and $x \neq 0$ leads to $\mathfrak{A} = \Phi x$, $\mathfrak{A}^2 = 0$. Thus $x = 0$. Also $\mathfrak{A}x = 0$ implies $x = 0$, by the same argument. Next, let c be a trivial element in $\mathfrak{A}$, $c\mathfrak{A}c = 0$. We consider the ideal $\mathfrak{A}c\mathfrak{A}$. Since $\mathfrak{A}c\mathfrak{A} = \mathfrak{A}$ leads to $\mathfrak{A}^2 = \mathfrak{A}c\mathfrak{A}\mathfrak{A}c\mathfrak{A} \subset \mathfrak{A}c\mathfrak{A}c\mathfrak{A} = 0$

we get $\mathfrak{A} c \mathfrak{A} = 0$. Then $\mathfrak{A}c = 0$ and $c = 0$, by the foregoing remark. Therefore $\mathfrak{A}$ has no trivial elements $\neq 0$ and the theorem applies.

8.2. Let V be a vectorspace over $\Phi = F$, F being a field, and $q : V \to F$ a quadratic form on V, i.e.,

$$q(\alpha x) = \alpha^2 q(x) \text{ for all } \alpha \in F,\ x \in V, \text{ and}$$

$$q(x,y) = \tfrac{1}{2}\left[q(x+y) - q(x) - q(y)\right] \text{ is bilinear (in } x \text{ and } y).$$

We wish to associate with (V,q) a Jordan algebra. The most obvious attempt will do it. We define

$$xy = q(x,y)1.$$

This, of course, is not a composition on V, but it leads to a composition on $\mathfrak{D} = F \oplus V$

$$\mathfrak{D} = F \cdot 1 \oplus V$$

if we define $(\alpha 1 + x)(\beta 1 + y) := (\alpha\beta + q(x,y))1 + \alpha y + \beta x$.

In particular, for $z = \alpha 1 + x$ we get

$z^2 = 2\alpha z + (\alpha^2 + q(x,x))1$, and furthermore 1 is unit element of $\mathfrak{D}$. This shows that the left multiplication $L(z^2)$ is a linear combination of $L(z)$ and $L(1) = \mathrm{id}$, which trivially implies $L(z)L(z^2) = L(z^2)L(z)$. Thus $\mathfrak{D}$ is a Jordan algebra. Exercise 1

Exercise 2: Show that $\mathfrak{D}$ is a quadratic extension of F or $\mathfrak{D} \cong F \oplus F$ if $\dim V = 1$ and q non degenerate. Now assume $\dim V > 2$.

Let $\mathfrak{U}$ be an ideal of $\mathfrak{D}$. If $\mathfrak{U} \cap V \neq 0$ and $z \neq 0$ is in this intersection, then by the nondegeneracy of q we can find a vector x such that $xu = q(x,u) = 1$. Since $xu \in \mathfrak{U}$, this shows $1 \in \mathfrak{U}$ and consequently $\mathfrak{U} = \mathfrak{D}$. Let $\rho 1 + v$ be a non zero element in $\mathfrak{U}$ and $\rho \neq 0$. Then for any vector $y \neq 0$, orthogonal to v ($\dim V \geq 2$), we