

Analysis in Several Variables Math 140C—Spring 2007

Complete Elegant Proof of the Chain Rule

THEOREM: If S and T are transformations such that T is differentiable at p_0 and S is differentiable at $T(p_0)$, then $S \circ T$ is differentiable at p_0 and

$$(S \circ T)'(p_0) = S'(T(p_0)) \circ T'(p_0).$$

PROOF:

Since S is differentiable at $T(p_0)$, we have

$$S(T(p)) - S(T(p_0)) - S'(T(p_0))(T(p) - T(p_0)) = B(p) \quad (1)$$

where $B(p)/|T(p) - T(p_0)| \rightarrow 0$ as $T(p) \rightarrow T(p_0)$. Since differentiability implies continuity, T is continuous at p_0 so that in fact

$$\frac{B(p)}{|T(p) - T(p_0)|} \rightarrow 0 \text{ as } p \rightarrow p_0. \quad (2)$$

Since T is differentiable at p_0 , we have

$$T(p) - T(p_0) - T'(p_0)(p - p_0) = C(p) \quad (3)$$

where

$$\frac{C(p)}{|p - p_0|} \rightarrow 0 \text{ as } p \rightarrow p_0. \quad (4)$$

Substitute (3) into (1) to get

$$S(T(p)) - S(T(p_0)) - S'(T(p_0))(T'(p_0)(p - p_0) + C(p)) = B(p).$$

and therefore

$$S(T(p)) - S(T(p_0)) - S'(T(p_0))T'(p_0)(p - p_0) = S'(T(p_0))C(p) + B(p). \quad (5)$$

Let $A(p) = S'(T(p_0))C(p) + B(p)$ denote the right side of (5). It remains to show that $A(p)/|p - p_0| \rightarrow 0$ as $p \rightarrow p_0$.

Since

$$\frac{A(p)}{|p - p_0|} = \frac{S'(T(p_0))C(p)}{|p - p_0|} + \frac{B(p)}{|p - p_0|} \quad (6)$$

and since $S'(T(p_0))$ is continuous, it follows from (4) that the first term on the right side of (6) approaches 0 as $p \rightarrow p_0$.

To show that the second term on the right side of (6) approaches 0 as $p \rightarrow p_0$, note first that $|T'(p_0)q| \leq M|q|$ for some constant M and all vectors q (continuity of linear transformations again), and therefore from (3)

$$|T(p) - T(p_0)| = |T'(p_0)(p - p_0) + C(p)| \leq |T'(p_0)(p - p_0)| + |C(p)| \leq M|p - p_0| + |C(p)|$$

so that

$$\frac{|B(p)|}{|T(p) - T(p_0)|} \geq \frac{|B(p)|}{M|p - p_0| + |C(p)|} = \frac{|B(p)|}{\left[M + \frac{|C(p)|}{|p - p_0|}\right] |p - p_0|},$$

and therefore from (2)

$$\frac{|B(p)|}{|p - p_0|} \leq \left[M + \frac{|C(p)|}{|p - p_0|}\right] \frac{|B(p)|}{|T(p) - T(p_0)|} \rightarrow M \cdot 0 = 0$$

as $p \rightarrow p_0$, as required. QED