VII. Linear Jordan Algebras.

7.1. Let Φ be a ~~commutative ring with unit element 1~~ field (either $\mathbb{R}$ or $\mathbb{C}$) containing $\frac{1}{2}$. An algebra $\mathfrak{J}$ over Φ with product $(x,y) \mapsto xy$ is called a <u>linear Jordan algebra</u>, if

(J.1) $\quad xy = yx \quad$ "commutativity"

(J.2) $\quad x(x^2y) = x^2(xy) \quad$ "Jordan identity"

for all $x,y \in \mathfrak{J}$.

In terms of the left and right multiplication $L(x), R(x)$, the above definition is obviously equivalent to

(J.1') $\quad L(x) = R(x)$

(J.2') $\quad L(x)L(x^2) = L(x^2)L(x) \quad$ for all $x \in \mathfrak{J}$.

<u>Example.</u> If $\mathfrak{A}$ is an associative algebra over Φ with product $(x,y) \mapsto xy$, then $\mathfrak{A}^+$, i.e., the module $\mathfrak{A}$ together with multiplication $(x,y) \mapsto x \circ y = \frac{1}{2}(xy + yx)$ is a Jordan algebra (see I, 1.1 , ex. 5). The powers of an element in $\mathfrak{A}^+$ are the same as in $\mathfrak{A}$. Furthermore, if $\mathfrak{A}$ has a unit element e, then e is also the unit element of $\mathfrak{A}^+$.

<u>Exercise.</u> 1 If $\mathfrak{J}$ is a Jordan algebra over Φ, then the unital algebra $\hat{\mathfrak{J}} = \Phi \cdot 1 \oplus \mathfrak{J}$ is again a Jordan algebra (see 1.7.).

7.2. A linearized form of the Jordan identity is (replace x by $x + \alpha z$, $\alpha = 1, \frac{1}{2}$).

(7.1) $\quad z(x^2y) + 2x((xz)y) = x^2(zy) + 2(xz)(xy)$

Linearizing again leads to (since we assume $\frac{1}{2} \in \Phi$)

(7.2) $\quad z((xu)y) + u((xz)y) + x((uz)y) = (xu)(zy) + (uz)(xy) + (xz)(uy)$.

This is in operator form (acting on z)

(7.3) $L(y(xu)) + L(u)L(y)L(x) + L(x)L(y)L(u) = L(xu)L(y) + L(uy)L(x) + L(xy)L(u)$.

Since the right hand side of this equation is symmetric in x and y we get

$L(y(xu)) + L(u)L(y)L(x) + L(x)L(y)L(u) = L(x(yu)) + L(u)L(x)L(y) + L(y)L(x)L(u)$,

or equivalently

(7.4) $L(x(yu) - y(xu)) = [[L(x), L(y)], L(u)]$

This equation has the following two interpretations

Lemma 1. The mappings $[L(x), L(y)]$, $x, y \in \mathfrak{J}$, are derivations of $\mathfrak{J}$. $\boxed{\text{Proof = Exercise 2}}$

Lemma 2. $L(\mathfrak{J})$ together with $(L(x), L(y), L(z)) \mapsto [[L(x), L(y)], L(z)]$ is a Lie triple system.

We denote by $\mathfrak{J}'$ the ~~submodule~~ vector subspace of $\mathfrak{J}$ spanned by all associators $(xy)z - x(yz)$, $x, y, z \in \mathfrak{J}$. Equation (7.4) shows that any Lie triple product of elements in $L(\mathfrak{J})$ is in $L(\mathfrak{J}')$, consequently, $L(\mathfrak{J}')$ is an ideal of $L(\mathfrak{J})$.

7.3. An important role in the theory of Jordan algebras plays the so-called quadratic representation P of a Jordan algebra $\mathfrak{J}$. This is a map $P : \mathfrak{J} \to \operatorname{End} \mathfrak{J}$, $x \mapsto P(x)$, defined by

(7.5) $P(x) = 2L(x)^2 - L(x^2)$, $x \in \mathfrak{J}$.

Note: $[L(x), L(x^2)] = 0$ implies $[L(x), P(x)] = 0$.

Example. If $\mathfrak{A}$ is associative, then the quadratic representation of $\mathfrak{A}^+$ is given by $P(x)y = xyx$.

The map P is quadratic in the sense that

$P(\alpha x) = \alpha^2 P(x)$ for all $\alpha \in \Phi$, $x \in \mathfrak{J}$, and

$P(x,y) := P(x+y) - P(x) - P(y)$ is bilinear (in x and y).

From the definition (7.5) we obtain easily

(7.6) $P(x,y) = 2\left[L(x)L(y) + L(y)L(x) - L(xy)\right]$, $P(x,x) = 2P(x)$,

Using (7.6) and (7.3) we compute

$P(xy,x) - L(y)P(x) - P(x)L(y) = 2L(xy)L(x) + 2L(x)L(xy) -$

$2L(x(xy)) - 2L(y)L(x)^2 + L(y)L(x^2) - 2L(x)^2L(y) + L(x^2)L(y)$

$= 2\left[L(x),L(xy)\right] + \left[L(y),\ L(x^2)\right] = 0$, since the last term is the linearized form of $\left[L(x),L(x^2)\right] = 0$, (J.2').

Consequently,

(7.7) $L(y)P(x) + P(x)L(y) = P(xy,x)$.

Furthermore we note that the linearization of $\left[L(x),P(x)\right] = 0$ is

(7.8) $\left[P(x,u),L(x)\right] = \left[L(u),P(x)\right]$.

An important composition in (linear) Jordan algebras is

$(x,y,z) \to \{xyz\} := P(x,z)y$.

This is obviously a trilinear composition, i.e., $\mathfrak{J}$ together with this composition is a triple system (see $\overline{III}$.). The "left multiplications" of this triple system are $L(x,y) \in \operatorname{End} \mathfrak{J}$, defined by

$$L(x,y)z = \{xyz\} = P(x,z)y$$

Using (7.6) we observe

$$L(x,y) = 2\left[L(x),L(y)\right] + 2L(xy).$$

Applying (7.7) repeatedly (and using $L(x)P(x) = P(x)L(x)$) we derive $\frac{1}{2}P(x)L(y,x) = P(x)L(y)L(x) - L(x)P(x)L(y) + P(x)L(xy) =$

$= [P(xy,x) - L(y)P(x)]\ L(x) - L(x)[P(xy,x) - L(y)P(x)] + P(x)L(xy)$

$= [L(x),L(y)]\ P(x) + [P(xy,x),L(x)] + P(x)L(xy)$

$= [L(x),L(y)]\ P(x) + L(xy)P(x)$ (by (7.8) with $u = xy$)

$= \frac{1}{2}L(x,y)P(x)$.

We proved $P(x)L(y,x) = L(x,y)P(x)$. Both sides of this equation acting on u shows $P(x)\{yxu\} = \{xyP(x)u\}$. Since the left hand side of the last equation is symmetric in y and u, we conclude $\{xyP(x)u\} = \{xuP(x)y\}$. This is in operator form $L(x,y)P(x) = P(P(x)y,x)$. We proved

(7.9) $L(x,y)P(x) = P(x)L(y,x) = P(P(x)y,x)$ "Homotopy formula".

The linearization of (7.7) acting on $v \in \mathcal{J}$ shows (after appropriate change of notation),

(7.10) $y \cdot \{uvw\} = \{(yu)vw\} - \{u(yu)w\} + \{uv(yw)\}$.

It is obvious from the definition, that for any derivation D of $\mathcal{J}$

$$D\{uvw\} = \{(Du)vw\} + \{u(Dv)w\} + \{uv(Dw)\}$$

holds. Then, in particular, this equation holds for $D = [L(x),L(y)]$, by lemma 1. Using this and (7.10) $(y \to xy)$, we derive

$L(x,y)\ \{uvw\} = 2[L(x),L(y)]\ \{uvw\} + 2L(xy)\ \{uvw\}$

$= \{(L(x,y)u)vw\} - \{u(L(y,x)v)w\} + \{uvL(x,y)w\}$.

This is

(7.11) $\{xy\{uvw\}\} - \{uv\{xyw\}\} = \{\{xyu\}vw\} - \{u\{yxv\}w\}$,

or in operator form

(7.11') $[L(x,y),L(u,v)] = L(\{xyu\},v) - L(u,\{yxv\})$.

A particular case of this equation is (setting $u = x$, $y = v$)

(7.12) $L(P(x)y,y) = L(x,P(y)x)$.

Furthermore we observe that the left hand side of (7.11) is skew symmetric in the pairs (x,y), (u,v), hence

$$(7.13) \qquad \{\{xyu\}vw\} - \{u\{yxv\}w\} = \{x\{vuy\}w\} - \{\{uvx\}yw\}.$$

In order to prove the <u>fundamental formula</u>

$$(7.14) \qquad P(P(u)v) = P(u)P(v)P(u) \quad \text{for all } u,v \in \mathcal{J},$$

we substitute $x \to \{uvu\}$, $w \to u$ in (7.11) and obtain (note: $\{xyx\} = 2P(x)y$)

$$(7.15) \qquad 8P(P(u)v)y = 2\{uv\{uy\{uvu\}\}\} - \{u\{y\{uvu\}\}u\}.$$

Replacing $u \to y$, $y \to u$, $x \to v$, $v \to u$, $w \to v$ in (7.13) gives

$$\{y\{uvu\}v\} = 2\{\{vuy\}uv\} - \{v\{uyu\}v\}.$$

Substituting this in (7.15) implies

$$8P(P(u)v)y = 2\{uv\{uy\{uvu\}\}\} - 2\{u\{\{uyv\}uv\}u\} + 8P(u)P(v)P(u)y.$$

Since the homotopy formula (7.9) has as consequence

$$\{uv\{uy\{uvu\}\}\} = \{uv\{u\{yuv\}u\}\} = \{u\{vu\{yuv\}\}u\},$$

the foregoing reduces to (7.14).

We have seen that the deduction from the axioms (J.1), (J.2) of all the important formulas in Jordan theory (in particular (7.9), (7.12) and (7.14)) depends heavily on the fact that we were able to cancel by 2. On the other hand, a theory of linear Jordan algebras over fields of characteristic 2 does not lead to results, which are "compatible" with results in the case of char $\neq 2$. So one has to think of something else, which would permit a "nice" theory for arbitrary rings. The best approach so far is via "quadratic Jordan algebras", which where "invented"