I. Nonassociative Algebras.

1.1. Let Φ be ~~a commutative ring with 1~~ the field $\mathbb{R}$ or $\mathbb{C}$. A ~~unitary Φ -module~~ vector space $\mathfrak{A}$ together with a bilinear map (multiplication) $\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$, $(a,b) \mapsto ab$, is called an algebra over Φ (or Φ-algebra). An algebra $\mathfrak{A}$ is called commutative if $ab = ba$ for all $a,b \in \mathfrak{A}$; it is called associative, if $(ab)c = a(bc)$ for all $a,b,c \in \mathfrak{A}$.

Examples. 1). Any ~~Φ-module~~ vector space $\mathfrak{A}$ together with $(a,b) \mapsto 0$ is an algebra.

2). If $\mathfrak{M}$ is a ~~Φ-module~~ vector space, then $\mathrm{End}_{\Phi}\mathfrak{M}$ together with the usual composition of mappings is an algebra, the algebra of endomorphisms of $\mathfrak{M}$. $\mathrm{End}_{\Phi}\mathfrak{M}$ is associative (but in general not commutative).

3). $\Phi^{(n,n)}$, the ~~Φ-module~~ vector space of $n \times n$ matrices over Φ together with usual matrix multiplication is an associative algebra.

4). In an associative algebra $\mathfrak{A}$ one often considers the commutator $[a,b] := ab - ba$. $\mathfrak{A}$ together with the map $(a,b) \mapsto [a,b]$ is an algebra, denoted by $\mathfrak{A}^-$.

Exercise 1

One easily checks

$$[a,a] = 0$$
$$[[a,b],c] + [[b,c],a] + [[c,a],b] = 0$$

for all $a,b,c \in \mathfrak{A}^-$.

5). If one considers the anticommutator $a \circ b := ab + ba$ in $\mathfrak{A}$ ($\mathfrak{A}$ associative) then $\mathfrak{A}$ together with $(a,b) \mapsto a \circ b$ is denoted by $\mathfrak{A}^+$ and one checks

$$a \circ b = b \circ a$$

$$a \circ ((a \circ a) \circ b) = (a \circ a) \circ (a \circ b) \quad \text{for all } a,b \in \mathfrak{A}^{+}.$$

An algebra $\mathfrak{L}$ (over Φ) is called a Lie algebra, if

(L.1) $\qquad xx = 0$

(L.2) $\qquad (xy)z + (yz)x + (zx)y = 0$ (Jacobi identity)

for all $x,y,z \in \mathfrak{L}$.

Example: If $\mathfrak{A}$ is associative then $\mathfrak{A}^{-}$ is a Lie algebra and any ~~submodule~~ subspace of $\mathfrak{A}$ closed under $[x,y]$ is a Lie algebra.

An algebra $\mathfrak{J}$ is called a Jordan algebra, if

(J.1) $\qquad xy = yx$

(J.2) $\qquad (xx)(xy) = x((xx)y)$

for all $x,y \in \mathfrak{J}$.

Example: Any ~~submodule~~ subspace of an associative algebra which is closed under $x \circ y$ is a Jordan algebra, in particular $\mathfrak{A}^{+}$ is a Jordan algebra.

1.2. Let $\mathfrak{A}$ be any nonassociative (that means not necessarily associative) Φ-algebra. For ~~submodules~~ subspaces $\mathfrak{U}, \mathfrak{V} \subset \mathfrak{A}$ we use the notations $\mathfrak{U}+\mathfrak{V}$ and $\mathfrak{U}\mathfrak{V}$ for the ~~submodules~~ subspaces generated by all $u + v$ rsp. uv, $u \in \mathfrak{U}, v \in \mathfrak{V}$. A ~~submodule~~ subspace $\mathfrak{U}$ is a subalgebra, if $\mathfrak{U}\mathfrak{U} \subset \mathfrak{U}$, it is an ideal, if $\mathfrak{A}\mathfrak{U} + \mathfrak{U}\mathfrak{A} \subset \mathfrak{U}$. An ideal $\mathfrak{Z}$ of $\mathfrak{A}$ is called a proper ideal, if $\mathfrak{Z} \neq 0$ and $\mathfrak{Z} \neq \mathfrak{A}$. $\mathfrak{A}$ is simple, if $\mathfrak{A}$ has no proper ideal and $\mathfrak{A}\mathfrak{A} \neq 0$.

If $\mathfrak{U}$ is an ideal in $\mathfrak{A}$, then one defines in a natural way in the quotient ~~module~~ vector space

$$\bar{\mathfrak{A}} = \mathfrak{A}/\mathfrak{U}$$

a multiplication

$$(a + \mathfrak{U})(b + \mathfrak{U}) := ab + \mathfrak{U} .$$

$\bar{\mathfrak{A}}$ together with this multiplication is called the <u>quotient algebra</u> of $\mathfrak{A}$ mod $\mathfrak{U}$.

A <u>homomorphism</u> of Φ-algebras $\mathfrak{A}$, $\mathfrak{A}'$ is a Φ-linear map $f: \mathfrak{A} \to \mathfrak{A}'$ such that $f(ab) = f(a)f(b)$ for all $a,b \in \mathfrak{A}$.

Isomorphisms and automorphisms are defined in the usual way.

We have the standard results.

<u>Theorem 1.</u> (i) <u>A subset $\mathfrak{L} \subset \mathfrak{A}$ is an ideal, iff $\mathfrak{L}$ is the kernel of some homomorphism.</u>

(ii) <u>If</u> $f: \mathfrak{A} \to \mathfrak{A}'$ <u>is a homomorphism (of algebras) then</u> $f(\mathfrak{A}) \cong \mathfrak{A}/\text{kernel } f$

(iii) <u>If</u> $\mathfrak{U}, \mathfrak{B}$ <u>are ideals in</u> $\mathfrak{A}$, <u>then</u>

$$\mathfrak{U} + \mathfrak{B}/\mathfrak{B} \cong \mathfrak{U}/\mathfrak{U} \cap \mathfrak{B} .$$

Exercise 2: Prove Theorem 1

1.3. Let $\mathfrak{A}$ be an algebra. A linear map $D: \mathfrak{A} \to \mathfrak{A}$ is called a <u>derivation</u> of $\mathfrak{A}$, if

$$D(ab) = (Da)b + a(Db) \quad \text{for all } a,b \in \mathfrak{A} .$$

One easily checks that for derivations D_1, D_2 the commutator $[D_1,D_2]$ again is a derivation, hence the subspace ~~Φ-module~~ of all derivations of $\mathfrak{A}$ together with the map $(D_1,D_2) \mapsto [D_1,D_2]$ is a Lie algebra (a subalgebra of $(\text{End}_\Phi \mathfrak{A})^-$). It is denoted by $\mathfrak{D}(\mathfrak{A}) = \mathcal{D}(\mathfrak{A})$ and called the <u>derivation algebra</u> of $\mathfrak{A}$.

1.4. For $a \in \mathfrak{A}$ we define endomorphisms $L(a)$ and $R(a)$ of $\mathfrak{A}$ by

$L(a): x \mapsto ax$; $R(a): x \mapsto xa$

i.e. $L(a)x = ax$, $R(a)x = xa$.

We call $L(a)$ rsp $R(a)$ the <u>left</u> (<u>rsp. right</u>) <u>multiplication</u> of a.

Exercise 3: Verify the assertions in (a)–(d) below

With these notations we rewrite some definitions.

a) $\mathfrak{A}$ is associative, i.e., $(xy)z = x(yz)$ for all $x,y,z \in \mathfrak{A}$ is equivalent to either

(i) $L(xy) = L(x)L(y)$

(ii) $R(yz) = R(z)R(y)$

(iii) $L(x)R(z) = R(z)L(x)$

(for all $x,y,z \in \mathfrak{A}$).

An easy computation shows that $L(x) - R(x)$, $x \in \mathfrak{A}$, are derivations of $\mathfrak{A}$.

b) Let $\mathcal{L}$ be a Lie algebra. In (L.1) we replace x by x + y and obtain $0 = (x + y)(x + y) = xx + xy + yx + yy = xy + yx$, or

$$xy = - yx.$$

with this the Jacobi identity may be written as

$$(xy)z = x(yz) - y(xz).$$

In terms of the left and right multiplications the last two equations are equivalent to

(L.1') $L(x) = - R(x)$

(L.2') $L(xy) = [L(x), L(y)]$

for all $x,y \in \mathcal{L}$.

c) Looking at ($\mathcal{J}$.1) and ($\mathcal{J}$.2) we see that an algebra $\mathfrak{J}$ is a Jordan algebra, iff

($\mathcal{J}$.1') $L(x) = R(x)$

($\mathcal{J}$.2') $L(x)L(xx) = L(xx)L(x)$

for all $x \in \mathfrak{J}$.

d) A linear map $D: \mathfrak{A} \to \mathfrak{A}$ is a derivation, iff

$$L(Dy) = [D, L(y)] \quad \text{for all } y \in \mathfrak{A}.$$

(L.2') shows that in a Lie algebra all left multiplications are derivations.

1.5. For any algebra $\mathfrak{A}$ one defines the "derived series"

$$\mathfrak{A} = \mathfrak{A}^{(0)} \supset \mathfrak{A}^{(1)} \supset \mathfrak{A}^{(2)} \ \ldots \ \supset \mathfrak{A}^{(k)} \ \ldots$$

by $\mathfrak{A}^{(0)} = \mathfrak{A}$, $\mathfrak{A}^{(k+1)} = \mathfrak{A}^{(k)}\mathfrak{A}^{(k)}$.

In general only $\mathfrak{A}^{(0)}$ and $\mathfrak{A}^{(1)}$ are ideals of $\mathfrak{A}$.

Exercise. 4 If $\mathcal{L}$ is a Lie algebra, then $\mathcal{L}^{(k)}$, $k \geq 0$, is an ideal of $\mathcal{L}$.

An algebra $\mathfrak{A}$ is called solvable, if $\mathfrak{A}^{(n)} = 0$ for some n.

Lemma 1. Subalgebras and homomorphic images of solvable algebras are solvable.

Proof. Easy exercise. 5

Lemma 2. If $\mathcal{B}$ is an ideal of $\mathfrak{A}$, then $\mathfrak{A}$ is solvable iff $\mathcal{B}$ and $\mathfrak{A}/\mathcal{B}$ are solvable.

Proof. One direction follows from lemma 1. By the definition of multiplication in $\mathfrak{A}/\mathcal{B}$ we get

$$\left(\mathfrak{A}/\mathcal{B}\right)^{(k)} = \mathfrak{A}^{(k)}/\mathcal{B}$$

The quotient being solvable implies $\mathfrak{A}^{(k)}/\mathcal{B} = 0$ for some k, or equivalently $\mathfrak{A}^{(k)} \subset \mathcal{B}$. But then

$\mathfrak{A}^{(k+s)} \subset (\mathfrak{A}^{(k)})^{(s)} \subset \mathcal{B}^{(s)} = 0$ for some s, since $\mathcal{B}$ is solvable. $\square$

Theorem 2. (i) If $\mathfrak{U}$, $\mathfrak{V}$ are solvable ideals in an algebra $\mathfrak{A}$, then $\mathfrak{U} + \mathfrak{V}$ is a solvable ideal.

(ii) If $\mathfrak{A}$ is Noetherian then $\mathfrak{A}$ has a unique maximal solvable ideal $\mathcal{R}(\mathfrak{A})$ which contains all other solvable ideals and furthermore $\mathcal{R}\left(\mathfrak{A}/\mathcal{R}(\mathfrak{A})\right) = 0$.

(Note: We call $\mathfrak{A}$ Noetherian, if every non-empty set of ideals has a maximal element.)

<u>Proof</u>. By theorem 1 (iii) we have

$$\mathfrak{U}+\mathfrak{B}/\mathfrak{B} \cong \mathfrak{U}/\mathfrak{U}\cap\mathfrak{B}.$$

Since $\mathfrak{U}/\mathfrak{U}\cap\mathfrak{B}$ is a homomorphic image of the solvable ideal $\mathfrak{U}$ it is solvable by lemma 1. Hence $\mathfrak{U}+\mathfrak{B}/\mathfrak{B}$ is solvable, and lemma 2 then shows that $\mathfrak{U}+\mathfrak{B}$ is solvable.

Let $\mathfrak{A}$ be Noetherian and $\mathfrak{R}(\mathfrak{A})$ a maximal element in the set of all solvable ideals in $\mathfrak{A}$ (this set contains the zero ideal). Let $\mathfrak{R}'$ be any solvable ideal; then $\mathfrak{R}(\mathfrak{A}) + \mathfrak{R}'$ is solvable by part (i) of the theorem. Since $\mathfrak{R}(\mathfrak{A}) \subset \mathfrak{R}(\mathfrak{A}) + \mathfrak{R}'$ we have $\mathfrak{R}(\mathfrak{A}) = \mathfrak{R}(\mathfrak{A}) + \mathfrak{R}'$ by the maximality of $\mathfrak{R}(\mathfrak{A})$. This implies $\mathfrak{R}' \subset \mathfrak{R}(\mathfrak{A})$ and if in particular $\mathfrak{R}'$ is maximal solvable then $\mathfrak{R}' = \mathfrak{R}(\mathfrak{A})$. If $\overline{\mathfrak{U}}$ is solvable in $\mathfrak{A}/\mathfrak{R}(\mathfrak{A})$, then $\mathfrak{U}$ is solvable in $\mathfrak{A}$, hence contained in $\mathfrak{R}(\mathfrak{A})$ and consequently $\overline{\mathfrak{U}} = 0$, which shows $\mathfrak{R}(\mathfrak{A}/\mathfrak{R}(\mathfrak{A})) = 0$. $\square$

The unique maximal solvable ideal $\mathfrak{R}(\mathfrak{A})$ is called the <u>solvable radical</u> of $\mathfrak{A}$.

1.6. Powers of an element $a \in \mathfrak{A}$ ($\mathfrak{A}$ an arbitrary algebra) are defined recursively by

$$a^1 = a, \quad a^{n+1} = a^n a.$$

In general $a^n a \neq a a^n$.

An algebra $\mathfrak{A}$ is called <u>power-associative</u>, if $a^n a^m = a^{n+m}$ for all $a \in \mathfrak{A}$, $n,m \geq 1$.

$a \in \mathfrak{A}$ is <u>nilpotent</u>, if $a^n = 0$ for some n. (0 is nilpotent).
An ideal $\mathfrak{B} \subset \mathfrak{A}$ is called <u>nil</u>, if all elements in $\mathfrak{B}$ are nilpotent.

<u>Lemma 3.</u> <u>Let $\mathfrak{A}$ be an algebra in which $(a^n)^m = a^{nm}$ for all $a \in \mathfrak{A}$, $n,m \geq 1$. If $\mathfrak{B}, \mathfrak{C}$ are nil ideals of $\mathfrak{A}$, then $\mathfrak{B} + \mathfrak{C}$ is nil.</u>

<u>Proof</u>. Let $b + c \in \mathfrak{B} + \mathfrak{C}$ ($b \in \mathfrak{B}$, $c \in \mathfrak{C}$), then
$(b + c)^n = b^n + d$ where $d \in \mathfrak{C}$. Since b is nilpotent we get
$(b + c)^n = d$ for some n. Since $d \in \mathfrak{C}$, it is nilpotent and with our assumption it follows
$(b + c)^{nm} = ((b + c)^n)^m = 0$ for some m. $\square$

Since the property of an ideal being nil is defined elementwise we get the existence of a maximal nil ideal $\mathfrak{N}(\mathfrak{A})$ by Zorn's lemma. The previous lemma shows that if $\mathfrak{A}$ is power associative then $\mathfrak{N}$ is uniquely determined; it is called the <u>nilradical</u> of $\mathfrak{A}$.

1.7. An element $e \in \mathfrak{A}$ (again $\mathfrak{A}$ arbitrary) is called a <u>unit element</u>, if $ea = ae = a$ for all $a \in \mathfrak{A}$, or equivalently, if $L(e) = R(e) = \mathrm{id}$, the identity mapping. $c \in \mathfrak{A}$ is called an <u>idempotent</u> of $\mathfrak{A}$ if $c \neq 0$ and $c^2 = c$.

There is a standard construction to imbed any algebra $\mathfrak{A}$ into an algebra $\hat{\mathfrak{A}}$ with unit element. Consider the ~~Φ-module~~ vector space

$$\hat{\mathfrak{A}} = \Phi \cdot 1 \oplus \mathfrak{A} = \{(\alpha, a);\ \alpha \in \Phi,\ a \in \mathfrak{A}\}$$

and define a multiplication in $\hat{\mathfrak{A}}$ by the formula

$$(\alpha, a)(\beta, b) := (\alpha\beta, \alpha b + \beta a + ab),$$

then $\hat{\mathfrak{A}}$ has a unit element $(1,0)$ and $a \mapsto (0,a)$ defines an isomorphism of $\mathfrak{A}$ into $\hat{\mathfrak{A}}$. By means of this isomorphism one identifies

$\mathfrak{A}$ with its image, so $\mathfrak{A}$ is an ideal in $\hat{\mathfrak{A}}$. Instead of $(\alpha, a) \in \hat{\mathfrak{A}}$ we write $\alpha 1 + a$. If $\mathfrak{A}$ is associative, so is $\hat{\mathfrak{A}}$ (easy exercise) 6 but if $\mathfrak{A}$ is a Lie algebra, $\hat{\mathfrak{A}}$ is not a Lie algebra, since a Lie algebra does not have a unit element $\neq 0$.

1.8. An endomorphism $j: \mathfrak{A} \to \mathfrak{A}$ of an algebra is called an <u>involution</u>, if

$$j(ab) = j(b)j(a)$$

$$j(j(a)) = a \qquad \text{for all } a,b \in \mathfrak{A}.$$

If $\mathfrak{A}^{op}$ denotes the algebra which has the same ~~module~~ vector space as $\mathfrak{A}$ but multiplication $(x,y) \mapsto x \circ y$ defined by $x \circ y = yx$ for all $x,y \in \mathfrak{A}$, then an involution may be viewed as a isomorphism $j: \mathfrak{A} \to \mathfrak{A}^{op}$. A ~~submodule~~ subspace $\mathcal{L} \subset \mathfrak{A}$ is j-stable if $j(\mathcal{L}) \subset \mathcal{L}$. Let $\mathfrak{A}$ have an involution j. The pair $(\mathfrak{A}, j)$ is called <u>simple</u>, if $\mathfrak{A}$ has no proper j-stable ideal and $\mathfrak{A}^2 \neq 0$.

<u>Theorem 3.</u> <u>Let $\mathfrak{A}$ be an algebra with involution j and $(\mathfrak{A}, j)$ simple. Then either</u>

(i) <u>$\mathfrak{A}$ is simple, or</u>

(ii) <u>$\mathfrak{A} \cong \mathcal{L} \oplus \mathcal{L}^{op}$, $\mathcal{L}$ a simple ideal of $\mathfrak{A}$ and</u>

$$j(b_1, b_2) = (b_2, b_1).$$

<u>Proof.</u> If $\mathfrak{A}$ is not simple, then it has a proper ideal $\mathcal{L}$. $(0 \neq \mathcal{L} \neq \mathfrak{A})$. It is obvious that $j(\mathcal{L}) \cap \mathcal{L}$ and $j(\mathcal{L}) + \mathcal{L}$ are j-stable ideals, consequently $j(\mathcal{L}) \cap \mathcal{L} = 0$ and $j(\mathcal{L}) \oplus \mathcal{L} = \mathfrak{A}$ since $(\mathfrak{A}, j)$ is simple. By the previous remarks $j(\mathcal{L})$ may be viewed as isomorphic image of $\mathcal{L}^{op}$, hence $\mathfrak{A} \cong \mathcal{L} \oplus \mathcal{L}^{op}$ and $j(b_1, b_2) = (b_2, b_1)$. If $\mathcal{J}$ is an ideal of $\mathcal{L}$ then $\mathfrak{A}\mathcal{J} \subset \mathcal{L}\mathcal{J}$,

since $\mathfrak{B}^{op}\mathfrak{C} \subset \mathfrak{B}^{op} \cap \mathfrak{B} = 0$. This shows that $\mathfrak{C}$ is an ideal in $\mathfrak{A}$ and if $\mathfrak{C} \neq 0$ the above construction shows $\mathfrak{A} = \mathfrak{C} \oplus j(\mathfrak{C})$. This implies $\mathfrak{B} = \mathfrak{C}$ and $\mathfrak{B}$ is simple. $\square$

1.9. An important tool in the structure theory of algebras are certain bilinear forms.

Let $\mathfrak{A}$ be an algebra over Φ and $\lambda : \mathfrak{A} \times \mathfrak{A} \to \Phi$ a bilinear form. λ is called <u>associative</u>, if

$$\lambda(xy,z) = \lambda(x,yz)$$

<u>Example</u>. If $\mathfrak{A}$ is a finite dimensional associative algebra over a field, then $(x,y) \mapsto \text{trace } L(xy)$ is an associative bilinear form.

The importance of such forms can be seen from

<u>Theorem 4</u>. <u>(Dieudonné)</u>. <u>Let $\mathfrak{A}$ be a finite dimensional algebra over a field F satisfying</u>

(i) <u>$\mathfrak{A}$ has a symmetric non degenerate associative bilinear form</u> λ,

(ii) <u>if $\mathfrak{B} \neq 0$ is an ideal of $\mathfrak{A}$, then $\mathfrak{B}^2 \neq 0$.</u>

<u>Then $\mathfrak{A}$ is a direct sum of simple ideals of $\mathfrak{A}$</u>.

<u>Proof</u>. Let $\mathfrak{B}$ be a minimal ideal ($\neq 0$) of $\mathfrak{A}$. The associativity of λ shows that $\mathfrak{B}^{\perp} = \{x, \ \lambda(x,\mathfrak{B}) = 0\}$ is an ideal of $\mathfrak{A}$. Since $\mathfrak{B} \cap \mathfrak{B}^{\perp}$ is an ideal in $\mathfrak{B}$, we get $\mathfrak{B} \cap \mathfrak{B}^{\perp} = \mathfrak{B}$ or $\mathfrak{B} \cap \mathfrak{B}^{\perp} = 0$, by the choice of $\mathfrak{B}$. Suppose the first case holds and let $b, b' \in \mathfrak{B}$, $a \in \mathfrak{A}$; then $0 = \lambda(ab,b') = \lambda(a,bb')$. Since λ is non degenerate, $bb' = 0$ and $\mathfrak{B}^2 = 0$, contrary to assumption. Hence $\mathfrak{B} \cap \mathfrak{B}^{\perp} = 0$ and $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{B}^{\perp}$ (Here we make use of the finite dimensionality of $\mathfrak{A}$). Any ideal of $\mathfrak{B}$ is an ideal of $\mathfrak{A}$ (same argument as in the proof of theorem 3), then by the minimality of

$\mathcal{L}$ it has no proper ideal. Since $\mathcal{L}^2 \neq 0$ by assumption, we see that $\mathcal{L}$ is simple. Since the assumptions (i) and (ii) are true in $\mathcal{L}^{\perp}$ we get by an induction argument the decomposition of $\mathfrak{A}$ as a direct sum of simple ideals. $\square$

II. Associative Algebras

2.1. Let $\mathfrak{A}$ be an associative algebra over a ring Φ and assume that $\mathfrak{A}$ has a unit element e. An element $a \in \mathfrak{A}$ is called <u>left invertible</u> (resp. <u>right invertible</u>) if there is an element $b \in \mathfrak{A}$ ($b' \in \mathfrak{A}$) such that $ba = e$ (resp. $ab' = e$). a is <u>invertible</u> if a is left and right invertible.

<u>Lemma 1.</u> <u>The following statements are equivalent,</u>

(i) $a \in \mathfrak{A}$ <u>is invertible</u>,

(ii) <u>there is a unique element</u> $a^{-1} \in \mathfrak{A}$ <u>such that</u> $a^{-1}a = aa^{-1} = e$

(iii) $L(a)$ <u>is invertible</u> (in $\operatorname{end}_{\Phi} \mathfrak{A}$).

<u>Proof.</u> Let $b, b' \in \mathfrak{A}$ be such that $ba = ab' = e$. Then $b = be = b(ab') = (ba)b' = eb' = b'$, consequently (i) $\to$ (ii). If $a^{-1}a = aa^{-1} = e$ then $L(a^{-1})L(a) = L(a)L(a^{-1}) = \mathrm{id}$. This shows that $L(a)$ is invertible and $L(a^{-1}) = L(a)^{-1}$, thus (ii) $\to$ (iii). To show (iii) $\to$ (i) assume $L(a)$ invertible, i.e. $L(a)U = UL(a) = \mathrm{id}$ for a unique $U \in \operatorname{end} \mathfrak{A}$ (apply (i) $\to$ (ii) to $\operatorname{End} \mathfrak{A}$). All terms of this equation acting on $e \in \mathfrak{A}$ gives $au = Ua = e$ for $u = Ue$. But then $L(a)L(u) = \mathrm{id}$