**MR566495 (82c:92013a)** 92A10 17D99
**Abraham, Victor M.**
**Linearizing quadratic transformations in genetic algeb**
*Proc. London Math. Soc. (3)* **40** (1980), *no. 2,* 346–363.



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**MR566496 (82c:92013b)** 92A10 17D99
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**The induced linear transformation in a genetic algebra.**
*Proc. London Math. Soc. (3)* **40** (1980), *no. 2,* 364–384.

In a genetic algebra $A$ the transformation $\varphi\colon x \to x^2$ is fundamental, representing the transition from a population $x$ (a distribution of genetic types) to its filial distribution under random mating. Other more general quadratic transformations occur, and one commonly wants to study the sequence $\varphi, \varphi^2, \varphi^3, \cdots$ which such a transformation generates. When $\varphi$ is $x \to x^2$, we are considering the sequence of so-called plenary powers $x^{[1]} = x$, $x^{[2]}, x^{[3]}, \cdots$ $(x^{[n]} = x^{[n-1]}x^{[n-1]})$. Generalizing a technique used by J. B. S. Haldane [J. Genetics **22** (1930), 359–372] and other geneticists (who did not use genetic algebras explicitly), P. Holgate [J. London Math. Soc. **42** (1967), 489–496; MR0218413 (36 #1499)] described a good tool for this purpose, called linearization. One replaces a quadratic transformation $\varphi$ $(A \to A)$ by a linear transformation $\tilde\varphi$ $(B \to B)$ in a vector space $B$ related to $A$ but of higher dimension. The induced space $B$ will have a basis consisting of coordinates which are monomial functions (not unique) of chosen coordinates in $A$.

In these two papers the author investigates this process of linearization more fully. His purpose in the first paper is described thus: (i) to find $\dim B$ in terms of $\dim A$ exactly, recursively, or asymptotically; (ii) to describe precisely, or to generate, the monomials required to form a basis of $B$. These problems are solved, first in simple cases, and then for the general Schafer genetic algebra [R. D. Schafer, Amer. J. Math. **71** (1949), 121–135; MR0027751 (10,350a); H. Gonshor, Proc. Edinburgh Math. Soc. (2) **17** (1970/71), 289–298; MR0302218 (46 #1371)]. Along the way he proves what he calls the first fundamental theorem of genetic algebra: the induced vector space $B$ is unique to within isomorphism. He considers especially those Schafer genetic algebras which require only quadratic functions of the coordinates to linearize the transformation $x \to x^2$, noticing that they have a simple structure. He also carries linearization outside the domain of genetic algebras in two ways: (a) by considering a generalization of Bernšteĭn algebras [Holgate, J. London Math. Soc. (2) **9** (1974/75), 612–623; MR0465270 (57 #5175)] and (b) by making the interesting observation that the class of algebras in which quadratic transformations may be linearized is larger than the class of Schafer genetic algebras. {The reviewer does not like the author's frequent use of the phrase "quadratic transformation" as if it necessarily meant $x \to x^2$.}

The second paper ranges widely around the concept of the induced transformation $\tilde\varphi$ described above. Here are some of the topics treated. (1) Theorem 1 ("the second fundamental theorem of genetic algebra"): A linearization of a quadratic transformation $x \to x^2$ in a given Schafer algebra is unique to within a similarity transformation. (2) "The fundamental equation of genetic algebra"—a formula for the $n$th iterate $x_n$ in terms of $x_0$ after $n$ generations of random mating under the usual simplifying assumptions. (3) "The inverse problem of genetic algebras": to find the initial population $x_0$, given the $n$th generation $x_n$. (4) A justification {was it needed?} of the reviewer's method of annulling polynomials [Quart. J. Math. Oxford Ser. (2) **12** (1941), 1–8; MR0005111 (3,102f)]. (5) Root vectors: If the minimal polynomial of $A$ is

$$\psi(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_s)^{m_s},$$

then the plenary train equation is

$$x(\varphi - \lambda_1)^{m_1}(\varphi - \lambda_2)^{m_2} \cdots (\varphi - \lambda_s)^{m_s} = 0.$$

Each factor $(\varphi - \lambda_j)^{m_j}$ makes a contribution $A_0 + A_1 n + A_2 n^2 + \cdots + A_{m_j - 1} n^{m_j - 1}$ to the solution $x_n$. The coefficients $A_i$ are called root vectors and they are studied at length. This involves generalized Bernoulli numbers. (6) Finally the author discusses limit theorems and the rate of convergence of $x_n$ to equilibrium, concluding (unsurprisingly) that the rate of convergence may be specified by giving the maximum multiplicity of a root in the plenary train equation and the maximum modulus of a root $\neq 1$.

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