Complex Analysis
Math 220C—Spring 2008

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Contents

1 Monday March 31, 2008—class cancelled due to the Master's travel plans

2 Wednesday April 2, 2008–Course information; Riemann's removable singularity theorem; Assignment 1

2.1 Course information

- Course: Mathematics 220C MWF 12:00–12:50 MSTB 114
- Instructor: Bernard Russo MSTB 263 Office Hours: Friday 2:30-4 or by appointment (Note: I am usually in my office the hour before class, that is, MWF 11)
- There is a link to this course on Russo's web page: www.math.uci.edu/∼brusso
- Homework: There will be assignments at almost every lecture with at least one week notice before the due date.
- Grading: TBA
- Holidays: May 26
- Text: Robert E. Greene and Steven G. Krantz, "Function Theory of One Complex Variable", third edition.
- Material to be Covered: We will cover chapter 10 and parts of chapters 9 and 15. We shall also provide a comprehensive review of complex analysis in preparation of the qualifying examination in complex analysis. The review will be based on notes from Math 220ABC (1993-94) which will be handed out. These notes follow the book by Conway (see below). Two copies of Conway have been put on one day reserve in the Science Library.
- Some alternate texts that you may want to look at, in no particular order. There are a great number of such texts at the undergraduate and at the graduate level.

  Undergraduate Level

  1. S. Fisher: Complex Variables
  2. R. Churchill and J. Brown; Complex Variables and Applications
  3. J. Marsden and M. Hoffman, Basic Complex Analysis
  4. E. Saff and A. Snider: Fundamentals of Complex Analysis

  Graduate Level

  1. L. Ahlfors; Complex Analysis
  2. J. Conway; Functions of one Complex Variable
  3. J. Bak and D. Newman; Complex Analysis

2.2 Riemann's removable singularity theorem

Theorem 2.1 (Riemann's Removable Singularity Theorem) *Let f be analytic on a punctured disk $B(a,R) - \{a\}$. Then f has an analytic extension to $B(a,R)$ if and only if $\lim_{z\to a}(z-a)f(z)$ exists and equals 0.*

Proof: If the analytic extension g exists, then $\lim_{z\to a}(z-a)f(z) = \lim_{z\to a}(z-a)g(z) = 0 \cdot g(a) = 0$.

Now suppose that $\lim_{z\to a}(z-a)f(z) = 0$. Define a function g by $g(z) = (z-a)f(z)$ for $z \neq a$ and $g(a) = 0$. The function g is analytic for $z \neq a$, and is continuous at a. We shall show using the Triangulated Morera theorem (see below) that g is analytic at a. Assuming for the moment that this is true, let us complete the proof. Since g is analytic and $g(a) = 0$, then by a consequence of the identity theorem, $g(z) = (z-a)h(z)$ where h is analytic in $B(a,R)$. Thus, for $z \neq a$, $(z-a)f(z) = g(z) = (z-a)h(z)$, and thus $f(z) = h(z)$ for $z \neq a$. Thus h is the analytic extension of f to $B(a,R)$.

It remains to prove that g is analytic at a. We first state a generalization of the theorem of Morera.

Theorem 2.2 (Triangulated Morera Theorem) *Let f be continuous on a domain D and suppose that $\int_T f(z)\,dz = 0$ for every triangle T which together with its inside lies in D. Then f is analytic in D.*

Proof: Let $a \in D$ and let $B(a,R) \subset D$. For $z \in B(a,R)$, let $F(z) := \int_{[a,z]} f(s)\,ds$ where $[a,z]$ denotes the line segment from a to z. For any other point $z_0 \in B(a,R)$, by our assumption, $F(z) = \int_{[a,z_0]} f(s)\,ds + \int_{[z_0,z]} f(s)\,ds$. Therefore

$$\frac{F(z)-F(z_0)}{z-z_0} - f(z_0) = \frac{1}{z-z_0}\int_{[z_0,z]} [f(s)-f(z_0)]\,ds$$

and

$$\left|\frac{F(z)-F(z_0)}{z-z_0} - f(z_0)\right| \leq \sup_{s\in[z_0,z]} |f(s)-f(z_0)|.$$

Since f is continuous at z_0, $F'(z_0)$ exists and equals $f(z_0)$ so f is analytic. □

To complete the proof of Riemann's removable singularity theorem, it remains to show that g is analytic using the Triangulated Morera theorem. We must show that if T is any triangle in $B(a,R)$, then $\int_T f(s)\,ds = 0$. There are four possible cases.

Case 1: <u>a **is a vertex of** T</u> : In this case let x and y denote points on the two edges for which a is an endpoint. Then $\int_T f(s)\,ds = \int_{[a,y,x]} f(s)\,ds + \int_{[y,x,b,c]} f(s)\,ds$ where b and c are the other two vertices of T and $[\alpha,\beta,\cdots]$ denotes a polygon with vertices $\alpha,\beta,\cdots$. By the continuity of g at a, the first integral approaches zero as x and y approach a. The second integral is zero by Cauchy's theorem.

Case 2: <u>a **is inside** T</u> : In this case, draw lines from a to each of the vertices of T. Then $\int_T f(s)\,ds$ is the sum of three integrals of f over triangles having a as a vertex, each of which is zero by case 1.

Case 3: <u>a **lies on an edge of** T</u> : In this case, draw a line from a to the vertex which is opposite to the edge containing a. Then $\int_T f(s)\,ds$ is the sum of two integrals of f over triangles having a as a vertex, each of which is zero by case 1.

Case 4: <u>a **is outside of** T</u> : In this case, $\int_T f(s)\,ds = 0$ by Cauchy's theorem. □

2.3 Examples of analytic continuations

Example 1 (Example 10.1.1 on page 299 of the text) Let $f(z) = \sum_0^\infty z^j$. The radius of convergence of this power series (a geometric series) is 1 and the sum is $1/(z-1)$. Thus the function f on $\{|z| < 1\}$ has an analytic continuation g (that is, an analytic extension) to $\mathbf{C} - \{1\}$, namely $g(z) = 1/(1-z)$, for $z \in \mathbf{C} - \{1\}$.

Example 2 (Example 10.1.2 on page 300 of the text) Let $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$ for $\operatorname{Re} z > 0$. Using integration by parts we can see that this holomorphic function continues analytically to $\mathbf{C} - \{0, -1, -2, \ldots\}$. The extension has poles of order 1 at each of the points $0, -1, -2, \ldots$ and the residue at $-k$ is $(-1)^k/k!$ (see Proposition 15.1.4 of the text—we will cover this later in the course)

Example 3(see section 15.2 of the text—we will cover this later in the course) Let $\zeta(z) = \sum_1^\infty n^{-z}$ which converges for $\operatorname{Re} z > 1$ to a holomorphic function, the zeta function. If we define $\eta(z) = \sum_1^\infty (-1)^{n+1}n^{-z}$ then it can be shown that this series converges for $\operatorname{Re} z > 0$ and it is easily seen that $\eta(z) = (1 - 2^{1-z})\zeta(z)$ holds for $\operatorname{Re} z > 1$. Using this formula as a definition, we see that ζ continues analytically to $\operatorname{Re} z > 0$. But in fact, because of the identity

$$\zeta(1-z) = 2^{1-z}\pi^{-z}\sin\left(\frac{1-z}{2}\pi\right)\Gamma(z)\zeta(z), \tag{1}$$

valid for $0 < \operatorname{Re} z < 1$, we can easily see that ζ analytically continues to $\mathbf{C} - \{1\}$. As a matter of fact, ζ has a pole at $z = 1$, and because of (1), it has zeros at the negative even integers. These are called the trivial zeros of the zeta function. The Riemann hypothesis is the (as yet unproven) assertion that all other zeros of the zeta function must lie on the line $\operatorname{Re} z = 1/2$.

Assignment 1 Problems 14 and 15 of chapter 6 of the text. These will de discussed in the informal discussion section on Friday April 4 at 4 pm in MSTB 256. The due date for this assignment is Friday April 11.

3 Friday April 4

3.1 Some definitions (Definitions 10.1.4 and 10.1.5 on page 303 and 10.2.1 on page 304 of Greene-Krantz)

function element This is an ordered pair (f, U), where U is a disc $D(P, r)$ and f is a holomorphic function defined on U.

direct analytic continuation The function element (g, V) is a *direct analytic continuation* of the function element (f, U) if $U \cap V \neq \emptyset$ and $f(z) = g(z)$ for all $z \in U \cap V$.

analytic continuation If (f_j, U_j) are function elements for $j = 1, \ldots k$ and if each (f_j, U_j) is a direct analytic continuation of (f_{j-1}, U_{j-1}), then we say that (f_k, U_k) is an *analytic continuation* of (f_1, U_1).

analytic continuation along a curve If (f, U) is a function element and U is a disc with center $\gamma(0)$ for some curve γ (parametrized by the unit interval $[0, 1]$), then an *analytic continuation* of (f, U) *along the curve* γ is a collection of function elements $\{(f_t, U_t) : t \in [0, 1]\}$ satisfying the following three properties:

1. $(f_0, U_0) = (f, U)$
2. For all $t \in [0,1]$, the center of the disc U_t is $\gamma(t)$
3. For all $t \in [0,1]$, there exists $\epsilon > 0$ such that for all $t_1 \in (t-\epsilon, t+\epsilon)$, we have
   (a) $\gamma(t_1) \in U_t$ (hence $U_{t_1} \cap U_t \neq \emptyset$)
   (b) $f_t(z) = f_{t_1}(z)$ for all $z \in U_{t_1} \cap U_t$ (hence (f_{t_1}, U_{t_1}) is a direct analytic continuation of (f_t, U_t).

Example: $\sqrt{z}$ (See Example 10.1.3 on page 301 of Greene-Krantz)

First of two uniqueness results:

Proposition 3.1 (Proposition 10.2.2 on page 305 of Greene-Krantz) *Any two analytic continuations of a function element along a curve are equivalent. This means that if* $\{(f_t, U_t) : t \in [0,1]\}$ *and* $\{(\tilde{f}_t, \tilde{U}_t) : t \in [0,1]\}$ *are analytic continuations of* (f, U) *along a curve* γ*, then* $f_t = \tilde{f}_t$ *on* $U_t \cap \tilde{U}_t$ *for all* $t \in [0,1]$.

3.2 Discussion section—Friday April 4

Problems 14 and 15 in chapter 6 of Greene-Krantz were discussed.

4 Monday April 7

Assignment 2 Exercises 9,10,11,12 of chapter 10 of Greene-Krantz. These will be discussed in the discussion section on Friday April 11 at 4 pm in MSTB 256. The due date for this assignment is Friday April 18.

4.1 The identity theorem and its proof

Theorem 4.1 *Let* D *be a connected open set and let* f *be analytic on* D*. The following are equivalent:*

(a) $f \equiv 0$*, that is,* $f(z) = 0$ *for every* z *in* D*.*

(b) *There exists a point* $z_0 \in D$ *such that* $f^{(n)}(z_0) = 0$ *for every* $n \geq 0$*.*

(c) *The set* $\{z \in D : f(z) = 0\}$ *has a limit point in* D*, that is, there is a sequence of* <u>*distinct*</u> *points* z_k *in* D *such that* $f(z_k) = 0$ *and* $\lim_{k\to\infty} z_k$ *exists and* <u>*belongs*</u> *to* D*.*

Proof: (a) implies (c) is trivial.

<u>(c) implies (b)</u>: Let z_0 be a limit point of $\{z \in D : f(z) = 0\}$ and suppose $z_0 \in D$. Since D is open, $\exists R > 0$ such that $B(z_0, R) \subset D$. Let us assume that (b) does not hold for any point of D. Then $\exists n \geq 1$ such that $0 = f(z_0) = f'(z_0) = \cdots = f^{(n-1)}(z_0)$ and $f^{(n)}(z_0) \neq 0$. Expanding f is a Taylor series about the point z_0, we have $f(z) = a_n(z-z_0)^n + a_{n+1}(z-z_0)^{n+1} + \cdots = (z-z_0)^n(a_n + a_{n+1}(z-z_0) + \cdots) = (z-z_0)^n g(z)$, where g is analytic and $g(z_0) = a_n = f^{(n)}(z_0)/n! \neq 0$. We have now reached a contradiction as follows. Since g is continuous and $g(z_0) \neq 0$, $\exists r,\ 0 < r \leq R$ with $g(z) \neq 0$ for $|z - z_0| < r$.

Hence $\{z \in D : f(z) = 0\} \cap B(z_0, r) = \{z_0\}$. This contradicts the fact that z_0 is a limit point of $\{z \in D : f(z) = 0\}$, and thus completes the proof of (c) implies (b).

(b) implies (a): Let $A = \{z \in D : \forall n \geq 0,\ f^{(n)}(z) = 0\}$. By assumption $A \neq \emptyset$. We shall prove that both $D - A$ and A are open sets. It will follow from the connectedness that $D = A$ and therefore f is identically zero in D.

A is open: Let $a \in A$. Since D is open, $\exists R > 0$ with $B(a, R) \subset D$. Write f in a Taylor series $f(z) = \sum_0^\infty a_n (z-a)^n$ for $|z - a| < R$ with $a_n = f^{(n)}(a)/n!$. Since $a \in A$, each $a_n = 0$ and so f is identically zero on $B(a, R)$. This means that $B(a, R) \subset A$ and so A is an open set.

$D - A$ is open: If $z \in D - A$, then there exists n_0 with $f^{(n_0)}(z) \neq 0$. Since $f^{(n_0)}$ is a continuous function, by "persistence of sign", there exists $r > 0$ such that $f^{(n_0)}$ never vanishes on $B(z, r)$. This says that $B(z, r) \subset D - A$ showing that $D - A$ is an open set. □

4.2 Proof of Proposition 3.1

(See Greene-Krantz, page 305-306). Let $S = \{t_0 \in [0, 1] : \forall t \leq t_0,\ f_t = \tilde{f}_t \text{ on } U_t \cap \tilde{U}_t\}$. Show that S is a closed set and is open relative to $[0, 1]$. (You use the identity theorem in the proofs of each of these statements.) Since $0 \in S$, the connectedness of $[0, 1]$ implies that $S = [0, 1]$, proving the proposition.

5 Wednesday April 9

5.1 Completion of the proof of Proposition 3.1

Namely, the proof that S is relatively open in $[0, 1]$ (Greene-Krantz, page 306).

5.2 The monodromy theorem

We now state the second of two uniqueness theorems concerning analytic continuation. In words, it states that if a function element admits analytic continuation along two homotopic curves, then the function elements at the end of each curve agree on the intersection of their domains. We first define some terms. (Definitions 10.3.1 and 10.3.2 on page 308 of Greene-Krantz)

homotopic curves Curves γ_0 and γ_1 in a connected open set $W \subset \mathbf{C}$ with $\gamma_0(0) = \gamma_1(0) = P$ and $\gamma_0(1) = \gamma_1(1) = Q$ are *homotopic in* W if there is a continuous function $H : [0, 1] \times [0, 1] \to W$ such that

1. $H(0, t) = \gamma_0(t)$ for all $t \in [0, 1]$
2. $H(1, t) = \gamma_1(t)$ for all $t \in [0, 1]$
3. $H(s, 0) = P$ for all $s \in [0, 1]$
4. $H(s, 1) = Q$ for all $s \in [0, 1]$

unrestricted continuation A function element (f, U) admits *unrestricted continuation in* W, an open connected set, if there is an analytic continuation of (f, U) along any curve beginning at the center of U and lying in W.

Theorem 5.1 (Theorem 10.3.3 on page 309 of Greene-Krantz) *Let $W \subset \mathbf{C}$ be a connected open set. Let (f, U) be a function element with $U \subset W$ and let P be the center of U. Assume that (f, U) admits unrestricted continuation in W. If γ_0 and γ_1 are curves in W that begin at P and have the same endpoint Q, then any analytic continuation along γ_0 agrees at Q with any analytic continuation along γ_1.*

6 Friday April 11

6.1 Proof of Theorem 5.1

(See Greene-Krantz, page 309-310). For each $s \in [0,1]$, let $\{(f_{s,t}, U_{s,t}) : t \in [0,1]\}$ be an analytic continuation of (f, U) along the curve $\gamma_s(t) = H(s,t)$.

Let $S = \{s \in [0,1] : \forall \lambda \le s,\ f_{\lambda,1} = f_{0,1} \text{ on } U_{\lambda,1} \cap U_{0,1}\}$. Show that S is an open set relative to $[0,1]$ and also a closed set. Since $0 \in S$, the connectedness of $[0,1]$ implies that $S = [0,1]$, proving the theorem[1].

6.2 Discussion section

Problems 9,10,11,12 in chapter 10 of Greene-Krantz were discussed.

7 Monday April 14

7.1 Hints for Assignment 1 (Problem 15)

1. The relation between two branches of the logarithm on a disc
2. The identity theorem
3. The assumed continuity of the function F on the unit circle.

7.2 Assignments 3 and 4

Assignment 3 (Due April 25)

Suppose f is analytic in $B(a,R) - \{a\}$ except for a sequence of poles $\{z_1, z_2, \ldots\}$ which converges to a. (Note that although f is not assumed to be analytic at a, nevertheless, a is not an isolated singularity of f.) Show that $f(B(a,R) - \{a, z_1, \ldots\})$ is dense in the complex plane.

> Hint: If it is not true, consider $g(z) = 1/(f(z) - w)$ for $z \in G := B(a,R) - \{a, z_1, z_2, \ldots\}$, where $w \in \mathbf{C}$ and $\delta > 0$ are such that $|f(z) - w| > \delta$ for all $z \in G$. Then obtain a contradiction by proving the following statements.

(a) There is an analytic extension h of g to $B(a,R) - \{a\}$ which vanishes at each z_n.

(b) The point a is an isolated singularity of h which is a removable singularity of h, and the analytic extension of h to a vanishes at a.

Assignment 4 (Due April 25) Problem 1 and one (your choice) of Problems 2,3,4 in Chapter 10 of Greene-Krantz.

[1] More details will be added here later

7.3 Notes for Math 220A, Fall 1993—for discussion on April 18 in the discussion section

The notes handed out cover the following material in Conway's book:

- complex numbers (in chapter 1)
- connectedness (in chapter 2)
- power series, logarithms, Cauchy-Riemann equations, Möbius transformations (in chapter 3)
- line integrals, integral and power series representations of analytic functions, consequences—Liouville, fundamental theorem of algebra, Cauchy estimates, maximum modulus (in chapter 4)
- maximum principle, Schwarz lemma (in chapter 6)

There are some 26 assignments in the notes. Attention could be focused on Assignments 5,6, 14–18, 21,25,26.

7.4 Context for Picard's (little) theorem

Liouville A non-constant entire function is unbounded.

Extended Liouville A non-constant entire function has dense range.

Casorati-Weierstrass The image of every neighborhood of an essential singularity has dense range.

Variation of Casorati-Weierstrass See Assignment 3

Little Picard A non-constant entire function maps onto the complement of a point in $\mathbf{C}$ (or $\mathbf{C}$).

Big Picard The image of every neighborhood of an essential singularity is the complement of a point in $\mathbf{C}$ (or $\mathbf{C}$).

8 Wednesday April 16

8.1 Assignments 5 and 6

An *automorphism* of an open set D in $\mathbf{C}$ is a holomorphic bijection of D. The inverse of an automorphism is automatically an automorphism. H is defined to be the group, under composition, of all automorphisms of the upper half plane $U = \{z \in \mathbf{C} : \mathrm{Im}\, z > 0\}$.

It follows from Assignment 6 below and the Cayley transform that

$$H = \{f(z) = \frac{az+b}{cz+d} : a, b, c, d \in \mathbf{R}, ad - bc > 0\}.$$

Also, for $f \in H$, we have

$$\mathrm{Im}\, f(z) = (ad - bc)\mathrm{Im}\, z.$$

Assignment 5 Due April 25

For a fixed complex number a with $|a| < 1$, define a function φ_a by

$$\varphi_a(z) = \frac{z+a}{1+\overline{a}z}.$$

Although $\varphi_a(z)$ is defined for all $z \neq -1/\overline{a}$, we shall consider it as a function on the closed unit disk $|z| \leq 1$. Prove the following statements.

(a) *If $|z| < 1$ then $|\varphi_a(z)| < 1$.*

(b) *If $|z| = 1$ then $|\varphi_a(z)| = 1$.*

(c) *φ_a is a one to one function, that is, if $|z_1| < 1, |z_2| < 1$ and if $\varphi_a(z_1) = \varphi_a(z_2)$, then $z_1 = z_2$.*

(d) *φ_a is an onto function, that is, if $|w_0| < 1$, then there is a z_0 with $|z_0| < 1$ and $\varphi_a(z_0) = w_0$.*

(e) *What is the inverse of φ_a?*

Assignment 6 Due April 25

Let f be an arbitrary analytic function on the unit disk $|z| < 1$ which is one to one and onto, that is, if $|z_1| < 1, |z_2| < 1$ and if $f(z_1) = f(z_2)$, then $z_1 = z_2$; and if $|w_0| < 1$, then there is a z_0 with $|z_0| < 1$ and $f(z_0) = w_0$. Prove the following statements.

(a) *If $f(0) = 0$, then $f(z) = e^{i\theta} z$ for some real θ.*

(b) *If $f(0) = a \neq 0$, let $g(z)$ be defined by $g(z) = \varphi_{-a}(f(z))$. Then $g(z) = e^{i\theta} z$ for some real θ.*

(c) *The function f has the form*

$$f(z) = e^{i\theta}\varphi_a(z),$$

for some θ real and $|a| < 1$.

8.2 The modular group

Let G be the subgroup of H defined as follows:

$$G = \{f \in H : f(z) = \frac{az+b}{cz+d} : a, b, c, d \in \mathbf{Z}, ad - bc = 1\}.$$

The *modular group*, denoted by Γ is the subgroup of G generated by the two elements

$$\mu(z) = \frac{z}{2z+1} \text{ and } \omega(z) = z + 2.$$

It is easy to see by induction that

$$\Gamma \subset \{f(z) = \frac{az+b}{cz+d} : a, d \text{ odd }, b, c \text{ even}, ad - bc = 1\}.$$

9 Friday April 18

9.1 Fundamental domain for the modular group

Because of the following propostion, the set W, defined as

$$W = \{z = x + iy : -1 \leq x < 1, |2z + 1| \geq 1, |2z - 1| > 1, y > 0\}$$

is called a *fundamental domain* for the action of the modular group Γ on the upper half plane.

Proposition 9.1 (Proposition 10.5.3 on page 318 of Greene-Krantz) *The modular group has the following properties:*

1. *If $f, g \in \Gamma$ and $f \neq g$, then $f(W) \cap g(W) = \emptyset$.*
2. *$U = \cup_{h \in \Gamma} h(W)$.*

9.2 Discussion section

This was the first of two discussion sections devoted to the notes for 220A Fall 1993. (The second one will be on April 25 at 3 pm or 4 pm.) We discussed the following problems in Conway: page 44, #13,17,21 and page 54, #8.

10 Monday April 21

10.1 Completion of the proof of Proposition 9.1

11 Wednesday April 23

11.1 The modular function

Theorem 11.1 (Theorem 10.5.4 on page 320 of Greene-Krantz) *There exists a holomorphic function λ on the open upper half plane U satisfying*

1. *$\lambda \circ h = \lambda$ for all $h \in \Gamma$*
2. *λ is one-to-one on W*
3. *$\lambda(U) = \mathbf{C} - \{0, 1\}$*
4. *λ has no analytic continuation to any open set strictly containing U*

The proof depends (among other things) on the following:

- Schwarz reflection principle (chapter 7)
- A version of Morera's theorem for smooth curves (Exercise 6 in chapter 3)
- Caratheodory extension theorem (chapter 13)
- Riemann mapping theorem (chapter 6)
- Uniqueness theorem (chapter 3)

12 Friday April 25

12.1 Discussion section—Friday April 25, 10:00 AM

We discussed Problem 1(a) on page 330 of Greene-Krantz. (Which is part of Assignment 4)

12.2 Discussion section—Friday April 25, 12:00 PM (instead of lecture)

We discussed the following problems in Conway's book: page 129, #3(a); page 132, #1,2,3.

13 Monday April 28

13.1 Picard's Little theorem

Theorem 13.1 (Theorem 10.5.5 on page 322 of Greene-Krantz) *An entire function f with $f(\mathbf{C}) \subset \mathbf{C} - \{0,1\}$ is a constant.*

Steps in the proof:

1. For each open disc $\mathcal{E} \subset \mathbf{C} - \{0,1\}$, there is an open subset $\mathcal{D}$ of the open upper half plane U such that $\lambda(\mathcal{D}) = \mathcal{E}$ and λ is one-to-one on $\mathcal{D}$. Hence, $(\rho, \mathcal{E})$ is a function element with values in U, where $\rho = (\lambda|\mathcal{D})^{-1}$. (The proof of this statement will be discussed on April 30 in class)

2. If $\mathcal{E}$ and $\mathcal{E}'$ are open discs contained in $\mathbf{C} - \{0,1\}$, and $\mathcal{E} \cap \mathcal{E}' \neq \emptyset$. Then, with obvious notation, $(\rho', \mathcal{E}')$ is a direct analytic continuation of $(\rho, \mathcal{E})$.

3. Let z_0 be any complex number and choose $\epsilon > 0$ so that $\mathcal{E}_0 := D(f(z_0), \epsilon) \subset \mathbf{C} - \{0,1\}$, and then by the continuity of f, choose $\delta > 0$ such that $\tilde{\mathcal{D}}_0 := D(z_0, \delta)$ satisfies $f(\tilde{\mathcal{D}}_0) \subset \mathcal{E}_0$. Now as in step 1, there is an open set $\mathcal{D}_0 \subset U$ and a function element $(\rho_0, \mathcal{E}_0)$, where ρ_0 is the inverse of $\lambda|\mathcal{D}_0$. Note that if we denote $\rho_0 \circ f$ by k_0, then $\lambda \circ k_0 = f$ on $\tilde{\mathcal{D}}_0$.

4. If γ is any curve in $\mathbf{C}$ starting at z_0, then $f \circ \gamma$ is a curve in $\mathbf{C} - \{0,1\}$ starting at $f(z_0)$. Therefore, for each $t \in [0,1]$ there is an open disc $\mathcal{E}_t \subset \mathbf{C} - \{0,1\}$ with center $f(\gamma(t))$, a disc $\tilde{\mathcal{D}}_t$ with center $\gamma(t)$ and an open set $\mathcal{D}_t \subset U$ such that $f(\tilde{\mathcal{D}}_t) \subset \mathcal{E}_t$, $(\rho_t, \mathcal{E}_t)$ is a function element with values in U and with ρ_t being the inverse of $\lambda|\mathcal{D}_t$, and $\lambda \circ k_t = f$ on $\tilde{\mathcal{D}}_t$ where $k_t := \rho_t \circ f$.

5. Using the compactness of the range of $f \circ \gamma$ you have a finite sequence of open discs $\mathcal{E}_i \subset \mathbf{C} - \{0,1\}$ with center $f(\gamma(t_i))$, a disc $\tilde{\mathcal{D}}_i$ with center $\gamma(t_i)$ and an open set $\mathcal{D}_i \subset U$ such that $f(\tilde{\mathcal{D}}_i) \subset \mathcal{E}_i$, $(\rho_i, \mathcal{E}_i)$ is a function element with values in U and with ρ_i being the inverse of $\lambda|\mathcal{D}_i$, and $\lambda \circ k_i = f$ on $\tilde{\mathcal{D}}_i$ where $k_i := \rho_i \circ f$.

   Moreover, $(\rho_i, \mathcal{E}_i)$ is a direct analytic continuation of $(\rho_{i-1}, \mathcal{E}_{i-1})$, and therefore (trivially since $f(\tilde{\mathcal{E}}_i) \subset \mathcal{E}_i$) $(\rho_i \circ f, \tilde{\mathcal{D}}_i)$ is a direct analytic continuation of $(\rho_{i-1} \circ f, \tilde{\mathcal{D}}_{i-1})$.

6. This shows that the function element $(\rho_0 \circ f, \tilde{\mathcal{D}}_0)$ can be analytically continued along any curve γ in $\mathbf{C}$.

7. Now we invoke the two uniqueness results about analytic continuation along curves, namely, Proposition 3.1 and Theorem 5.1.

8. Since **C** is simply connected, any two curves with the same beginning and same end point are homotopic. Thus the function element $(\rho_0 \circ f, \tilde{\mathcal{D}}_0)$ extends to an entire function $\mathcal{K}$ with $\mathcal{K}(\mathbf{C}) \subset U$.

9. Of course, $\mathcal{K}$ must be a constant.

10. Recall that $\lambda \circ k_0 = f$ on $\tilde{\mathcal{D}}_0$, where $k_0 = \rho_0 \circ f$. Thus $\lambda \circ \mathcal{K} = f$ on $\tilde{\mathcal{D}}_0$. So by the uniqueness theorem, $\lambda \circ \mathcal{K} = f$ on $\mathbf{C}$. Since $\mathcal{K}$ is a constant, so is f. □

14 Wednesday April 30

14.1 Handouts

1. Notes for 220B Winter 1994. These notes and the assignments therein will be the basis for the class meetings during weeks 7 and 8. The notes for 220C Spring 1994, which will be handed out later, will be the basis for the class meetings during weeks 9 and 10.

2. Exercises from Conway; pages 43,54,73,80,129,132. These will be discussed further during week 6 (=next week). Assignment 8 below will be from these exercises.

14.2 Discussion of Assignment 1

Page 203, Problem 14

(a) and (b): Use $f(z) = \log |z| + i\arg_\alpha$ where α is any real number such that the ray emanating from the origin making an angle α with the positive real axis does not intersect the given disc and $\alpha < \arg_\alpha < \alpha + 2\pi$. Or use a general result from the text on simple connectedness (Lemma 6.6.4, page 197)

(c) Assuming f is a branch of the logarithm, note that from $e^{f(z)} = z$ you get $f'(z) = 1/z$ so that $\int_{|z|=1} f'(z)\, dz = 0$ by the fundamental theorem of calculus, and $\int_{|z|=1} f'(z)\, dz = 2\pi i$ by direct calculation, a contradiction. Another way to prove this is to note that f must agree with one of the branches in part (a) and (b) above (except for a constant factor) and so by the uniqueness theorem must agree with that branch on the intersection of U and the domain of that branch. This is a contradiction since f is continuous on $|z| = 1$ and the branch is not.

(d) As in (c), g agrees, up to a constant factor, with one of those branches on any disc and is therefore differentiable.

Page 203, Problem 15

The hints for this problem were given above (see section 7.1). These ideas were used in some of the solutions above to the previous problem. That is all I will say.

14.3 Discussion of the first step in the proof of Picard's little theorem

Let $W' = \{z \in W : \mathrm{Re}\, z > 0\}$. By the Riemann mapping theorem, there is a holomorphic bijection

$$g_1 : W' \to U$$

of W' and the upper half plance U. We are going to assume without proof the theorem of Caratheodory (Theorem 13.2.3 on page 393). By this theorem, there is a continuous bijection

$$g_2 : \overline{W'} \to \overline{U},$$

extending g_1 with $g_2(0) = 0$ and $g_2(1) = 1$. Applying the Schwarz reflection theorem (Theorem 7.5.2, page 222), first to g_1 and then to g_2 you obtain a holomorphic bijection

$$g_3 : W^O \to \mathbf{C} - \{z = x \geq 0\}$$

extending g_1 and a continuous surjection

$$g_4 : \overline{W} \to \mathbf{C}$$

extending g_3. Finally, we let

$$g_5 : W \to \mathbf{C} - \{0, 1\}$$

be the restriction of g_4 to W, which is a continuous bijection.

Now let us recall the the modular function $\lambda : U - rightarrow \mathbf{C} - \{0, 1\}$ was defined to be equal to g_5 on W, and $g_5 \circ h^{-1}$ on $h(W)$. (See Proposition 9.1 and Theorem 11.1.)

We can now start to address the first step in the proof of Picard's little theorem. Let $\mathcal{E}$ be an open disc contained in $\mathbf{C} - \{0, 1\}$.

Case 1: $\mathcal{E} \subset U$. In this case, take $\mathcal{D} = g_1^{-1}(\mathcal{E})$.

Case 2: $\mathcal{E} \subset \mathbf{C} - \overline{U}$. In this case, take $\mathcal{D} = g_3^{-1}(\mathcal{E})$.

Case 3: $\mathcal{E} \cap \mathbf{R} \neq \emptyset$. This is the next assignment (also worth \$10)

Assignment 7 Due on May 16. Complete the proof of Picard's little theorem (Case 3).

15 Friday May 2

15.1 Discussion and answers to Assignment 2

Just the answers:

Page 331, Problem 9

An indefinite integral of $-[\text{Log}(1 - z)]/z$ on $\mathbf{C} - \{z = x \geq 1\}$

Page 331, Problem 10

$-\text{Log}(1 - z)$ on $\mathbf{C} - \{z = x \geq 1\}$

Page 331, Problem 11

Originally defined on $\{-1 < \text{Re}\, z < 3\}$. Extends to a meromorphic function on $\{\text{Re}\, z < 3\}$ with poles at $\{-1, -5, -9, -`3, \ldots\}$. Successive integrations by parts gives you the closed forms on each of the domains $\{-k < \text{Re}\, z < 3\}$, where $k = 5, 9, 13, \ldots$. Besides integration by parts, you use the elementary facts that the improper integral $\int_0^a t^p \, dt$ converges if $p > -1$ and the improper integral $\int_a^\infty t^p \, dt$ converges if $p < -1$.

Page 331, Problem 12

$-[\text{Log}(1 - z^2) - z^2]/z^2$ on $\mathbf{C} - \{z = x, |x| \geq 1\}$.

15.2 Discussion of Assignment 3

Extend g to a continuous function h on $B(a,R) - \{a\}$ by setting $h(z_j) = 0$ for each j. By Riemann's removable singularity theorem, h is holomorphic on $B(a,R) - \{a\}$. Again by Riemann's removable singularity theorem, h extends to a holomorphic function on $B(a,R)$. But $h(z_j) = 0$ so $h(a) = 0$ and h is identically zero by the uniqueness theorem, contradiction.

15.3 An assignment based on the 220A notes, 1993

Assignment 8 Due May 12. You should review all the problems below and write up any two of them from each page (total of 10 problems)

Conway's book

page 43: #9,11,12,13,14,15,16,17,18,19,20,21 (ANY TWO)
page 54: #8,9,18,19,24 (ANY TWO)
page 80:#1,4,5,6,7,8,9,10 (ANY TWO)
page 129:#1,2,3,4,5,6,7,8 (ANY TWO)
page 132:#1,2,3,4,5,6,7,8 (ANY TWO)

15.4 Discussion section did not meet today

16 Monday May 5

16.1 Automorphisms of the unit disc, revisited

The proof of the structure of the group of automorphisms of the unit disc using the derivative part of Schwarz's lemma is given on pages 131-132 of Conway. The proof in Assignment 6 uses a different part of Schwarz's lemma.

16.2 Some corrections and hints for Conway, page 132

I pointed out a correction to Conway's problem 1 on page 132. See page 19 of my notes for Math 220C Spring 1994 (Be careful, there are two page 19's). Also given on that page are some hints for Problems 5 and 6 on page 132 of Conway. Page 29 of the same notes gives hints for Problems 4 and more on Problem 5 on page 132 of Conway. These notes will be handed out on Friday.

17 Wednesday May 7

17.1 Linear fractional transformations

This material is only sparsely covered in the text by Greene-Krantz (see section 6.3, page 184). A large number of important properties is given by exercises 26-31 in Greene-Krantz (page 205), whose solutions can be found in Conway's book (Propositions 3.8,3.9,3.10,3.14,3.19) on pages 47–54. Copies of these 8 pages will also be handed out on Friday. Conway also considers orientation in his Theorem 3.21 on page 53.

Most of this class's time was spent discussing Problem 18 on page 55 of Conway, which is concerned with orientation.

18 Friday May 9

18.1 Discussion of Assignment 4—analytic continuation

We discussed Problems 1(a) and 1(b) on page 330, which is part of Assignment 4.

18.2 Two handouts

The Math220C notes for Spring1994 were handed out as well as copies of pages 47–54 of Conway's book.

18.3 Real integrals using residue calculus

We reviewed the 5 examples in Greene-Krantz on pages 128–136 and tried to identify which exercises among #46–58 on page 145 fall into which category of example.

18.4 Discussion section 3 pm in MSTB 256

We discussed Problems 8,9, and 24. For #24, we didn't use the hint given by Conway. Instead, we used the fact that if a linear fractional transformation S has two *distinct* fixed points, say a and b, then it has the form

$$\frac{Sz-a}{Sz-b} = k\frac{z-a}{z-b}$$

for some constant k. This useful fact can was found on page 86 of the book "Complex Analysis" by Lars Ahlfors (see the list of books I gave at the beginning of this course).

19 Monday May 12

19.1 Discussion of Assignment #4 (problem 3 in chapter 10)

19.2 Continuation of the discussion of real integrals using residues—Greene&Krantz page 145

19.3 Exam schedule

The complex analysis qualifying examination will take place on Wednesday June 18 10 AM-12:30 PM

The final exam in Mathematics 220C will take place on Monday June 9, 1:30-3:30 PM

Some materials for you to use in preparing for these exams:

- The 220A final examination for Fall 1993 (see the minutes of Math 220B Winter 1994)
- The 220B final examination for Winter 1994 (see the minutes of Math 220C Spring 1994)
- Assignment 8 in this course, corresponding to the minutes of Math 220A Fall 1993)
- Assignment 9 in this course, corresponding to the minutes of Math 220B Winter 1994)
- Assignment 10 in this course, corresponding to the minutes of Math 220C Spring 1994)

- Previous qualifying exams in complex analysis—see Paul Macklin's web page (there is a link on Professor Gorodetski's web page)

20 Wednesday May 14

20.1 Discussion of the problems for Math 220B Winter 1994

We discussed the problems on pages 83,87,95,99, and 110.

Assignment 9 Due May 30. You should review all the problems below and write up any 10 of them.

Conway's book

page 83: #3,4
page 87: #9
page 95; #10,11
page 99: #2,4
page 110; #1,6,10,13,15,17
page 121:# 2cdefgh (two of these)
page 126:# 2, 9,10

21 Friday May 16

21.1 Discussion of the problems on page 126 of Conway

21.2 Discussion section 3 PM

We discussed #9 on page 126 in detail.

22 Monday May 19

22.1 Discussion of the results of Assignment 8

The papers were handed back; and the exercises for Assignment 10 were handed out.

Assignment 10 Due June 9. You should review all the problems below and write up any 10 of them.

Conway's book

page 163: #2,3,5,6,7
page 173: #3,9,11,12,13
page 255; #4,5,6,8,9,11
(for #11, Theorem 1.6 is the maximum principle for harmonic functions)
page 262: #4(a,b),5,6
page 268; We are skipping this; enough is enough (subharmonic functions)

23 Wednesday May 21—no class

24 Friday May 23—no class

25 Monday May 26—holiday

26 Wednesday May 28—no class

27 Friday May 30

Assignment #9 collected

27.1 Discussion of the problems on page 163 of Conway (Riemann mapping theorem)

We discussed #4 and #9 (which was not solved) and made some cryptic remarks about #2 (can involve maximum and minimum modulus theorems), #5 (Schwarz lemma) and #6 and #7 (uniqueness of the Riemann map).

27.2 Discussion of the problems on page 173 of Conway (Weierstrass factorization)

During the class we discussed #5,7 and made some cryptic remarks about #1,4,10 (4(b) is written up in Greene-Krantz §9.1) and #12,13 (simply apply the Weierstrass theorem). In the 3 pm discussion section, we discussed #11 in detail as well as #4(c).

28 Monday June 2

28.1 Discussion of problems in Conway, page 255 (harmonic functions)

28.2 Discussion of problems in Conway, page 262 (harmonic functions)

29 Wednesday June 4

29.1 Discussion of the results of Assignment 9 (returned today)

30 Friday June 6