Elementary Analysis Math 140B—Winter 2007
Homework answers—Assignment 8; February 7, 2007

Exercise 25.12, page 191

Suppose that $\sum_{k=1}^{\infty} g_k$ is a series of continuous functions g_k on $[a, b]$ that converges uniformly to g on $[a, b]$. Prove that

$$\int_a^b g(x)\,dx = \sum_{k=1}^{\infty} \int_a^b g_k(x)\,dx.$$

Solution: Use Theorem 25.2 page 185.

$$\begin{aligned}
\int_a^b g(x)\,dx &= \int_a^b \left[\sum_{k=1}^{\infty} g_k(x)\right] dx = \int_a^b \left[\lim_{n\to\infty} \sum_{k=1}^{n} g_k(x)\right] dx \\
&= \lim_{n\to\infty} \int_a^b \left[\sum_{k=1}^{n} g_k(x)\right] dx = \lim_{n\to\infty} \sum_{k=1}^{n} \int_a^b g_k(x)\,dx \\
&= \sum_{k=1}^{\infty} \int_a^b g_k(x)\,dx.
\end{aligned}$$

Exercise 25.5, page 190

Assume that f_n converges uniformly to f on S and that each f_n is bounded on S. Prove that f is a bounded function on S.

Solution: For any $\epsilon > 0$, choose N so that $|f_n(x) - f(x)| < \epsilon$ for $n > N$ and $x \in S$. Then for a fixed $n > N$ and all $x \in S$,

$$|f(x)| \le |f(x) - f_n(x)| + |f_n(x)| < \epsilon + M_n,$$

where $M_n = \sup_{x\in S} |f_n(x)|$. Thus $\sup_{x\in S} |f(x)| \le \epsilon + M_n$ so f is bounded.

Exercise 27.3(a), page 204

Show that there does not exist a sequence of polynomials converging uniformly on $\mathbf{R}$ to $f(x) = \sin x$.

Solution: Assume that a polynomial p satisfies $|p(x) - \sin x| < 1$ for all $x \in \mathbf{R}$. Clearly, p cannot be a constant function. (proof: if $p(x) = c$, then $|c \pm 1| < 1$ which implies $c = 0$ and therefore $|\sin x| < 1$ for all x, a contradiction.) But a non-constant polynomial is unbounded. (proof: write $p(x) = x^n[a_n + a_{n-1}/x + a_{n-2}/x^2 + \cdots + a_0/x^n]$, with $a_n \neq 0$; then $\lim_{x\to\infty} |p(x)| = \infty \cdot [a_n + 0] = \infty$.) This contradicts the following:

$$|p(x)| \le |p(x) - \sin x| + |\sin x| \le 1 + 1 = 2.$$