Elementary Analysis Math 140B—Winter 2007
Homework answers—Assignment 26; March 19, 2007

Exercise 32.6

Let f be a bounded function on $[a,b]$. Suppose there exist sequences (U_n) and (L_n) of upper and lower Darboux sums for f such that $\lim(U_n - L_n) = 0$. Show f is integrable and $\int_a^b f = \lim U_n = \lim L_n$.

Solution: We are given that $U_n = U(f, P_n)$ and $L_n = L(f, Q_n)$ for certain partitions P_n and Q_n of $[a,b]$. Since

$$U(f, P_n \cup Q_n) - L(f, P_n \cup Q_n) \le U_n - L_n \to 0$$

it follows that for any $\epsilon > 0$, there is a partition $P_\epsilon = P_n \cup Q_n$ for some n such that

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon.$$

Hence f is integrable on $[a,b]$.

From $L_n \le L(f) = U(f) = \int_a^b f \le U_n$ we have $0 \le \int_a^b f - L_n \le U_n - L_n$ and $L_n - U_n \le \int_a^b f \le 0$ and therefore $\lim_n L_n = \lim_n U_n = \int_a^b f$.

Exercise 33.8

Let f and g be integrable functions on $[a,b]$

(a) Show that fg is integrable on $[a,b]$.

Solution: Since $4fg = (f+g)^2 - (f-g)^2$ and Exercise 33.7 states that the square of an integrable function is integrable, using linearity (Theorem 33.3) it follows that fg is integrable.

(b) Show that $\max(f,g)$ and $\min(f,g)$ are integrable on $[a,b]$.

Solution: Since $\min(f,g) = (f+g)/2 - |f-g|/2$ and $\max(f,g) = -\max(-f,-g)$, using Theorem 33.5 and linearity (Theorem 33.3), it follows that $\max(f,g)$ and $\min(f,g)$ are integrable.

Exercise 34.2

Calculate

(a) $\lim_{x\to 0} \frac{1}{x} \int_0^x e^{t^2}\, dt$

Solution: Let $F(x) = \int_0^x e^{t^2}\, dt$. Then $\lim_{x\to 0} \frac{1}{x} \int_0^x e^{t^2}\, dt = \lim_{x\to 0} \frac{F(x)}{x} = F'(0) = e^{x^2}|_{x=0} = 1$.

(b) $\lim_{h\to 0} \frac{1}{h} \int_3^{3+h} e^{t^2}\, dt$

Solution: Let $g(x) = \int_3^x e^{t^2}\, dt$ so that $g(3) = 0$ and $g'(x) = e^{x^2}$. Then $\lim_{h\to 0} \frac{1}{h} \int_3^{3+h} e^{t^2}\, dt = \lim_{h\to 0} \frac{g(3+h)-g(3)}{h} = g'(3) = e^9$.

Exercise 34.9

Use Example 3 to show $\int_0^{1/2} \sin^{-1} x\, dx = \pi/12 + \sqrt{3}/2 - 1$
Solution: Take $a = 0, b = \pi/6$ and $g(x) = \sin x$ in Example 3. Then

$$\int_0^{\pi/6} \sin x\, dx + \int_0^{1/2} \sin^{-1} x\, dx = \frac{\pi}{6} \cdot \frac{1}{2} = \frac{\pi}{12}$$

and

$$\int_0^{\pi/6} \sin x\, dx = -\cos x\, |_0^{\pi/6} = -\frac{\sqrt{3}}{2} + 1.$$