Elementary Analysis Math 140B—Winter 2007
Homework answers—Assignment 20; March 16, 2007

Exercise 31.4

Consider $a, b \in \mathbf{R}$ where $a < b$. Show that there exist infinitely differentiable functions $f_a, g_b, h_{a,b}$ and $h^*_{a,b}$ with the following properties.

(a) $f_a(x) = 0$ for $x \le a$ and $f_a(x) > 0$ for $x > a$

Solution: Let $f_a(x) = f(x - a)$, where f is the function in Example 3 on page 238, namely, $f(x) = 0$ for $x \le 0$ and $f(x) = e^{-1/x}$ for $x > 0$. We know that f is infinitely differentiable and therefore so is f_a.

(b) $g_b(x) = 0$ for $x \ge b$ and $g_b(x) > 0$ for $x < b$.

Solution: Taking the hint from (a), let $g_b(x) = e^{1/(x-b)}$ for $x < b$ and $g_b(x) = 0$ for $x \ge b$.

(c) $h_{a,b}(x) > 0$ for $x \in (a, b)$ and $h_{a,b}(x) = 0$ for $x \notin (a, b)$.

Solution: $h_{a,b} = f_a(x)g_b(x)$

(d) $h^*_{a,b}(x) = 0$ for $x \le a$ and $h^*_{a,b}(x) = 1$ for $x \ge b$.

Solution: $h^*_{a.b}(x) = f_a(x)/(f_a(x) + g_b(x))$

Exercise 31.6

A standard proof of Theorem 31.3 goes as follows. Assume $x > 0$, let M be as in the proof of Theorem 31.3, that is, such that

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + \frac{Mx^n}{n!}$$

and for $t \in [0, x]$, let

$$F(t) := f(t) + \sum_{k=1}^{n-1} \frac{(x-t)^k}{k!} f^{(k)}(t) + M \cdot \frac{(x-t)^n}{n!}.$$

(a) Show that F is differentiable on $[0, x]$ and that

$$F'(t) = \frac{(x-t)^{n-1}}{(n-1)!}[f^{(n)}(t) - M].$$

Solution:

$$F'(t) = f'(t) + \sum_{k=1}^{n-1} \left[\frac{(x-t)^k}{k!} f^{(k+1)}(t) - \frac{(x-t)^{k-1}}{(k-1)!} f^{(k)}(t) \right] - M\frac{(x-t)^{n-1}}{(n-1)!} = \frac{(x-t)^{n-1}}{(n-1)!}[f^{(n)}(t) - M].$$

(b) Show that $F(0) = F(x)$

Solution: $F(x) = f(x)$ and $F(0) = f(0) + \sum_{k=1}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + \frac{Mx^n}{n!}$ so by the defintion of M, $F(x) = F(0)$.

(c) Apply Rolle's theorem 29.2 to F to obtain y in $(0, x)$ such that $f^{(n)}(y) = M$.

Solution: Rolle's theorem (or the Mean Value Theorem) gives you a $y \in (0, x)$ such that

$$0 = \frac{F(x) - F(0)}{x} = F'(y) = \frac{(x-y)^{n-1}}{(n-1)!}[f^{(n)}(y) - M].$$

Thus $f^{(n)}(y) = M$, as required.