

Elementary Analysis Math 140B—Winter 2007
Homework answers—Assignment 19; March 12, 2007

Exercise 30.2

$$(a) \quad \lim_{x \rightarrow 0} \frac{x^3}{\sin x - x} = \lim_{x \rightarrow 0} \frac{3x^2}{\cos x - 1} = \lim_{x \rightarrow 0} \frac{6x}{-\sin x} = -6.$$

$$(b) \quad \lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} = \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x \rightarrow 0} \frac{2 \sec^2 x \tan x}{6x} = 2 \lim_{x \rightarrow 0} \sec^2 x \lim_{x \rightarrow 0} \frac{\sin x}{x} \lim_{x \rightarrow 0} \frac{1}{6 \cos x} = 2 \cdot 1 \cdot 1 \cdot \frac{1}{6} = \frac{1}{3}.$$

$$(c) \quad \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \rightarrow 0} \frac{x - \sin x}{x \sin x} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x \cos x + \sin x} = \lim_{x \rightarrow 0} \frac{\sin x}{-x \sin x + 2 \cos x} = 0.$$

(d) $\cos x$ is positive for x near 0; hence

$$\lim_{x \rightarrow 0} \frac{\log \cos x}{x^2} = \lim_{x \rightarrow 0} \frac{-\frac{\sin x}{\cos x}}{2x} = - \lim_{x \rightarrow 0} \frac{1}{2} \frac{1}{\cos x} \frac{\sin x}{x} = -\frac{1}{2}.$$

Hence

$$\lim_{x \rightarrow 0} (\cos x)^{1/x^2} = e^{-1/2}.$$

Exercise 30.3

$$(a) \quad \lim_{x \rightarrow \infty} \frac{x - \sin x}{x} = \lim_{x \rightarrow \infty} \left(1 - \frac{\sin x}{x} \right) = 1 - 0 = 1.$$

$$(b) \quad \sin \frac{1}{x} \log x = -\sin \frac{1}{x} \log \frac{1}{x} \text{ so if } t = \frac{1}{x} \text{ we have}$$

$$\lim_{t \rightarrow 0+} \sin t \log t = \lim_{t \rightarrow 0+} \frac{\frac{\sin t}{t}}{t \log t} = \frac{1}{\lim_{t \rightarrow 0+} t \log t},$$

and

$$\lim_{t \rightarrow 0+} t \log t = \lim_{t \rightarrow 0+} \frac{\log t}{1/t} = \lim_{t \rightarrow 0+} \frac{1/t}{-1/t^2} = \lim_{t \rightarrow 0+} (-t) = 0$$

from the left. Hence

$$\lim_{x \rightarrow \infty} \sin \frac{1}{x} \log x = \frac{-1}{-0} = +\infty.$$

Therefore $\lim_{x \rightarrow \infty} x^{\sin(1/x)} = e^\infty = \infty$.

$$(c) \quad \lim_{x \rightarrow 0+} \frac{1 + \cos x}{e^x - 1} = \frac{2}{0} = \infty.$$

$$(d) \quad \lim_{x \rightarrow 0} \frac{1 - \cos 2x - 2x^2}{x^4} = \lim_{x \rightarrow 0} \frac{2 \sin 2x - 4x}{4x^3} = \lim_{x \rightarrow 0} \frac{4 \cos 2x - 4}{12x^2} = \lim_{x \rightarrow 0} \frac{-8 \sin 2x}{24x} = \lim_{x \rightarrow 0} \frac{-8 \sin 2x}{12 \cdot 2x} = -\frac{2}{3}.$$

Exercise 30.4 Let f be a function defined on some interval $(0, a)$ and define $g(y) = f(1/y)$ for $y \in (a^{-1}, \infty)$. Show that $\lim_{x \rightarrow 0+} f(x)$ exists if and only if $\lim_{y \rightarrow \infty} g(y)$ exists, in which case they are equal.

Solution:

\Rightarrow : Suppose that $L := \lim_{x \rightarrow 0+} f(x)$ exists. Then given $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < x < \delta$. Let $M = 1/\delta$. Then for $y > M$, we have $0 < 1/y < 1/M = \delta$, so $|f(1/y) - L| < \epsilon$, proving that $\lim_{y \rightarrow \infty} g(y)$ exists, and equals L .

\Rightarrow : Suppose that $\lim_{y \rightarrow \infty} g(y)$ exists. Then given $\epsilon > 0$ there exists $M > 0$ such that $|g(y) - L| < \epsilon$ whenever $y > M$. Let $\delta = 1/M$. Then for $0 < x < \delta$, we have $1/x > 1/\delta = M$, so $|g(1/x) - L| < \epsilon$, proving that $\lim_{x \rightarrow 0+} f(x)$ exists and equals L .

Exercise 30.6 Let f be differentiable on some interval (c, ∞) and suppose that $\lim_{x \rightarrow \infty} [f(x) + f'(x)] = L$. Prove that $\lim_{x \rightarrow \infty} f(x) = L$ and that $\lim_{x \rightarrow \infty} f'(x) = 0$.

Solution: $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{f(x)e^x}{e^x} = \lim_{x \rightarrow \infty} \frac{f(x)e^x + f'(x)e^x}{e^x} = \lim_{x \rightarrow \infty} [f(x) + f'(x)] = L$, since $e^z \rightarrow \infty$. (It doesn't matter that $\lim_{x \rightarrow \infty} f(x)e^x$ might not exist.)