Elementary Analysis Math 140B—Winter 2007
Homework answers—Assignment 18; February 28, 2007

Use the intermediate theorem for derivatives to prove that a derivative cannot have a jump discontinuity. More precisely, if $f'(x)$ exists for all x in an open interval (a, b), if $y \in (a, b)$ and if

$$f'(y+) := \lim_{x \to y+} f'(x) \text{ and } f'(y-) := \lim_{x \to y-} f'(x) \text{ exist,}$$

then f' is continuous at y.

Solution: We show first that $f'(y-) = f'(y)$.

Case 1: Assume $f'(y-) < f'(y)$.

Choose N so that for $n \geq N$,

$$f'(y-) + 1/n < f'(y) - 1/n.$$

Now choose $x_n \in (y - 1/n, y)$ such that $f'(x_n) < f'(y-) + 1/n$. By the intermediate value theorem for derivatives, there exists $z_n \in [x_n, y]$ such that $f'(z_n) = f'(y) - 1/n$. Passing to the limit in the last equality gives $f'(y-) = f'(y)$, which is a contradiction to what we assumed in this case. So Case 1 does not occur.

Case 2: Assume $f'(y-) > f'(y)$.

Choose N so that for $n \geq N$,

$$f'(y-) - 1/n > f'(y) + 1/n.$$

Now choose $x_n \in (y - 1/n, y)$ such that $f'(x_n) > f'(y-) - 1/n$. By the intermediate value theorem for derivatives, there exists $z_n \in [x_n, y]$ such that $f'(z_n) = f'(y) + 1/n$. Passing to the limit in the last equality gives $f'(y-) = f'(y)$, which is a contradiction to what we assumed in this case. So Case 2 does not occur.

Since Case 1 and Case 2 are impossible, we have proved that $f'(y-) = f'(y)$.

By an entirely similar argument, it can be shown that $f'(y+) = f'(y)$. □