Elementary Analysis Math 140B—Winter 2007
Homework answers—Assignment 10; February 19, 2007

Exercise 28.4, page 211

Let $f(x) = x^2 \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$.

(a) Show that f is differentiable at each $a \neq 0$ and calculate $f'(a)$. **Solution:** If $x \neq 0$, then

$$f'(x) = x^2 \left(\cos \frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) + 2x \sin \frac{1}{x} = -\cos \frac{1}{x} + 2x \sin \frac{1}{x}.$$

(b) Show that f is differentiable at $x = 0$ and that $f'(0) = 0$.

Solution: For $x \neq 0$,

$$\frac{f(x) - f(0)}{x} = x \sin \frac{1}{x} \to 0 \text{ as } x \to 0.$$

(c) Show that f' is not continuous at 0.

Solution: If f' were continuous at 0, then we would have $0 = \lim_{x\to 0} f'(x)$ and therefore

$$\lim_{x\to 0} \cos \frac{1}{x} = \lim_{x\to 0} \left[2x \sin \frac{1}{x} - f'(x)\right] = 0,$$

a contradiction, since $\cos \frac{1}{x}$ oscillates as $x \to 0$.

Exercise 28.8, page 212

Let $f(x) = x^2$ for x rational and $f(x) = 0$ for x irrational.

(a) Prove that f is continuous at $x = 0$.

Solution: Given $\epsilon > 0$, let $\delta = \sqrt{\epsilon}$. Then

$$|f(x) - f(0)| = |f(x)| \leq x^2 \text{ so if } |x| < \delta \text{ then } |f(x) - f(0)| < \epsilon.$$

(b) Prove that f is discontinuous at all $x \neq 0$.

Solution: If $x_0 \neq 0$, let $\epsilon = x_0^2/2$ which is > 0. Assume that there is a $\delta > 0$ so that $|f(x_0) - f(x)| < \epsilon$ for $|x - x_0| < \delta$. If x_0 is rational, pick x irrational such that $|x - x_0| < \delta$. Then $|f(x_0) - f(x)| = x_0^2 > \epsilon$, a contradiction.

On the other hand, if x_0 is irrational, first pick $\delta' < \delta$ such that $|x^2 - x_0^2| < \epsilon$ if $|x - x_0| < \delta'$. This is possible since the function $g(x) = x^2$ is continuous everywhere. Now pick x rational such that $|x - x_0| < \delta'$. Then $|x_0^2 - x^2| < \epsilon$ by continuity of g and $x^2 < \epsilon$ by the supposed continuity of f. Hence $x_0^2 - x^2 \leq |x_0^2 - x^2| < \epsilon = x_0^2/2$, and $x_0^2/2 \leq x^2 < \epsilon = x_0^2/2$, a contradiction.

(c) Prove that f is differentiable at $x = 0$.

Solution:

$$\frac{f(x) - f(0)}{x} = \frac{f(x)}{x} \text{ so that } \left|\frac{f(x) - f(0)}{x}\right| \leq \frac{x^2}{|x|} = |x| \to 0$$

as $x \to 0$. Note that $f'(0) = 0$.