Introduction to differential equations

Sept. 29, 2008

First-order linear differential equations

Def

$y' + a(t)y = b(t)$, $y(t)$

the first order linear (non homogeneous) differential equation

$y' + a(t)y = 0$

the first order linear homogeneous diff. equation

Example

$y' + ty + \sin y = 0$ X

$y' + ty + \sin t = 0$ ✓

How to solve

$y' + a(t)y = 0$ (*)

$\frac{y'}{y} = -a(t)$

$\frac{d}{dt} \ln |y| = -a(t)$

$\ln |y| = -\int a(t)dt + \tilde{c}$

$|y(t)| = e^{-\int a(t)dt} \cdot e^{\tilde{c}}$

$\boxed{y(t) = c \cdot e^{-\int a(t)dt}}$ general solution of (*)

Example

$$y' = ty \qquad \left(\underbrace{ce^{t^2/2}}_{y}\right)' = \underbrace{c\cdot e^{t^2/2}}_{y}\cdot t$$

$$\frac{y'}{y} = t$$

$$y(t) = ce^{t^2/2}$$

How to solve $y' + a(t)y = b(t)$? $(**)$

Solve first $y' + a(t)y = 0$

$$y(t) = C\cdot e^{-\int a(t)dt}$$

Let us try to find a solution of $(**)$ in a form

$$y(t) = \varphi(t)\cdot e^{-\int a(t)dt}$$

$$\varphi'(t)e^{-\int a(t)dt} + \cancel{\varphi(t)\cdot e^{-\int a(t)dt}\cdot(-a(t))} + \cancel{a(t)\varphi(t)e^{-\int a(t)dt}} = b(t)$$

$$\varphi'(t) = b(t)\cdot e^{\int a(t)dt}$$

$$\varphi(t) = C + \int b(t)e^{\int a(t)dt}dt$$

$$\varphi(t) = C + G(t)$$

$$y(t) = e^{-\int a(t)dt}\left(C + G(t)\right)$$

(Method of „variation of parameters")

Example

$$y' + y + t = 0$$

$$y' = -y$$

$$y = Ce^{-t}$$

Let us try to find the solution of the (nonhomogeneous) equation in a form

$$y(t) = \varphi(t) \cdot e^{-t}$$

$$\varphi'(t)e^{-t} - \cancel{\varphi(t) \cdot e^{-t}} + \cancel{\varphi(t)e^{-t}} + t = 0$$

$$\varphi'(t) = -te^{t}$$

$$\varphi(t) = -\int te^{t}dt + C = -(te^{t} - e^{t}) + C =$$

$$= C - te^{t} + e^{t}$$

$$y(t) = (C - te^{t} + e^{t})e^{-t} = \underbrace{Ce^{-t}} - t + 1$$

– general solution

Remark

Solutions of a homogeneous linear diff. equation form a linear space:

$$y' + a(t)y = 0 \qquad y = y(t)$$

$$z' + a(t)z = 0 \qquad z = z(t)$$

$$\frac{d}{dt}(y+z) + a(t)(y+z) = 0$$

$$(\lambda y)' + a(t) \cdot (\lambda y) = 0 \qquad \forall \lambda \in \mathbb{R}.$$

Moreover, if

z, y are solutions of a non-homogeneous equation then

$(z-y)$ is a solution of a homogeneous equation.

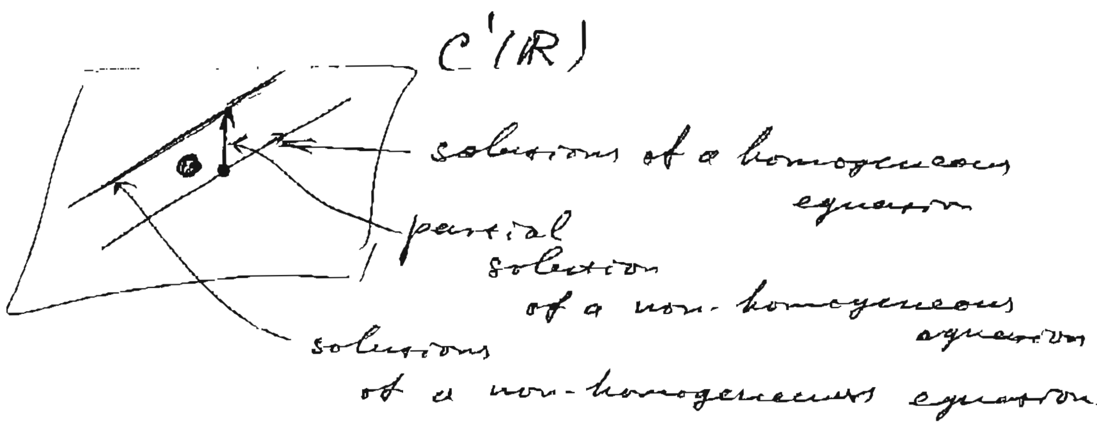
In other words, if

$w(t)$ is a solution of a non-homogeneous equation,

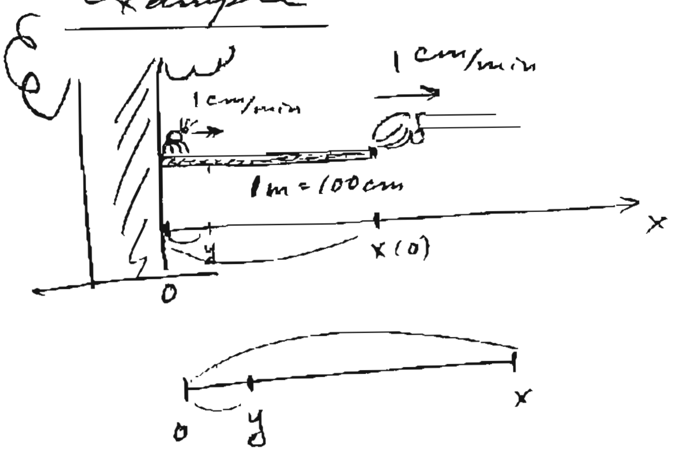
then any its solution $z(t)$

can be represented as $z(t) = w(t) + u(t)$,

where $u(t)$ is a solution of a homogeneous equation



Example



$x(t) = 100 + t$

$$\boxed{\begin{aligned} &y'(t) = 1 + \frac{y}{x} = 1 + \frac{y}{100+t} \\ &y(0) = 0 \end{aligned}}$$

Can the bug bite within 2 hours?

Homogeneous equation:

$$y'(t) = \frac{y}{100+t}$$

$$\frac{y'}{y} = \frac{1}{100+t}$$

$$y = C(100+t)$$

Non-homogeneous equation:

$$y' = 1 + \frac{y}{100+t}$$

$$y(t) = \varphi(t)\cdot(100+t)$$

$$\varphi'(t)\cdot(100+t) + \cancel{\varphi(t)} = 1 + \cancel{\varphi(t)}$$

$$\varphi'(t) = \frac{1}{100+t}$$

$$\varphi(t) = \ln(100+t) + C$$

$$y(t) = (100+t)(C + \ln(100+t))$$

$$y(0) = 0$$

$$0 = (100+t)(C + \ln 100)$$

$$C = -\ln 100$$

$$\boxed{y(t) = (100+t)\ln\left(1+\frac{t}{100}\right)}$$

$$y(t) = X(t) = 100+t$$

$$\ln\left(1+\frac{t}{100}\right) = 1 \Rightarrow 1+\frac{t}{100} = e$$

$$t = 100(e-1) \simeq 1.7\cdot 10^2 \text{ min}$$

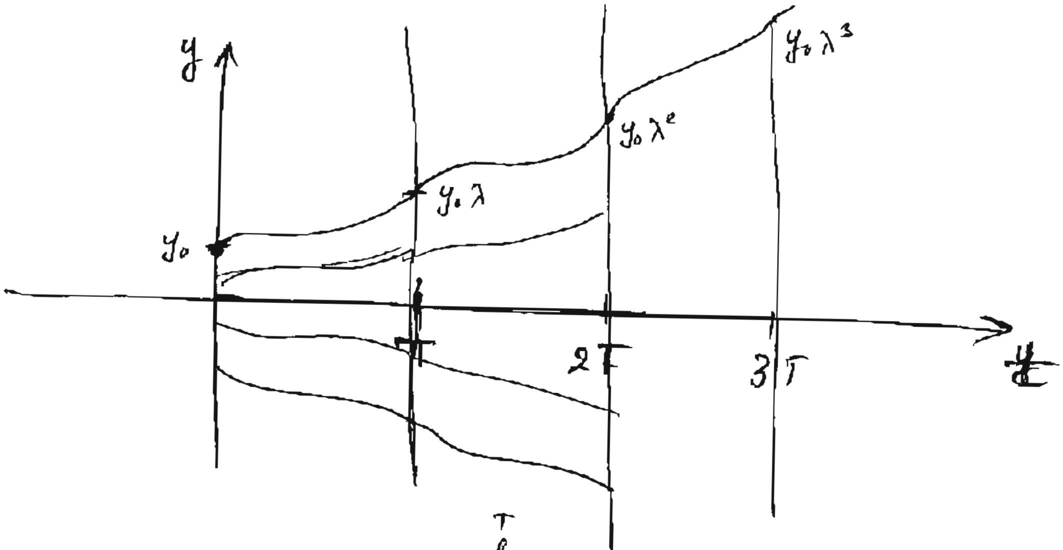
170 min > 2 hours !

On linear diff. equations of first order with periodic coefficients.

$y'(t) + a(t)y = 0$, $a(t+T) \equiv a(t)$, $T > 0$.

$y(t) = Ce^{-\int a(t)dt}$

$y(T) = y(0)\underbrace{e^{-\int_0^T a(t)dt}}_{=\lambda}$



If $\lambda = e^{-\int_0^T a(t)dt} < 1$ then $y(t) \to 0$ as $t \to +\infty$

If $\lambda > 1$ then $y(t) \to \infty$ as $t \to +\infty$.

If $\lambda = 1$ then $y(t)$ is periodic.

$y' + a(t)y = b(t)$ - $a(t)$, $b(t)$ are periodic with period T.

If $\lambda = e^{-\int_0^T a(t)dt} \neq 1$ then $\exists!$ periodic solution,

if $\lambda < 1$ then all solutions $\to$ to periodic as $t \to +\infty$,

if $\lambda > 1$ then all other solutions $\to \infty$ as $t \to +\infty$.