Introduction to differential equations

Lecture 1 (Sept. 26, 2008)

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textbooks: M. Braun - required
Arnol'd ODE - recommended

Differential equations

Ordinary differential equations (ODE)

"Find a function $y(t)$ such that

$$F(t, y, y', \dots, y^{(n)}) = 0"$$

Partial differential equations (PDE)

"Find a function $u(t_1, t_2, \dots, t_n)$ such that

$$F\left(t, u, \frac{\partial u}{\partial t_1}, \dots, \frac{\partial u}{\partial t_k}, \dots \frac{\partial^n u}{\partial t_1^{k_1} \cdots \partial t_k^{n_k}}\right) = 0"$$

Example:

$$y' = 2y$$

$$y'' + ty + \cos t = 0$$

$$y''' = y + t, \quad y = y(t)$$

$$\frac{\partial^2 u(t,x)}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad u = u(t,$$

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} = 0$$

$$u = u(x_1, x_2, x_3$$

Definition

The Order of a differential equation is the order of the highest derivative of the function y that appears in the equation

Solution of a differential equation:

$y(t)$ is a solution of ODE if it satisfies the equation.

Example

$(*)\quad y' = y \qquad y(t) = e^t$ - solution $\quad (e^t)' = e^t$

$y(t) = 2e^t$ - solution $\quad (2e^t)' = 2e^t$

$y(t) = Ce^t$ - solution of $(*)$ for each $C \in \mathbb{R}$.

↑ general solution of $(*)$

To find a general solution of a differential equation ~~is~~ to find all solutions.

Initial value problem:

find a solution $y(t)$ of a differential equation such that

$$y(t_0) = y_0,\ y'(t_0) = y_1,\ \ldots,\ y^{(n-1)}(t_0) = y_{n-1},$$

where n is an order of the diff. equation

Example

Find a solution of an equation

$$y' = y$$

such that $y(0) = 1$.

Solution: $y(t) = Ce^t$

If $y(0) = 1$, then $C \cdot e^0 = C = 1$, so we are looking for a solution $y(t) = e^t$ □

Simplest differential equation.

$$y'(t) = f(t) \qquad (**)$$

General solution:

$$y(t) = \int f(t)\,dt + C$$

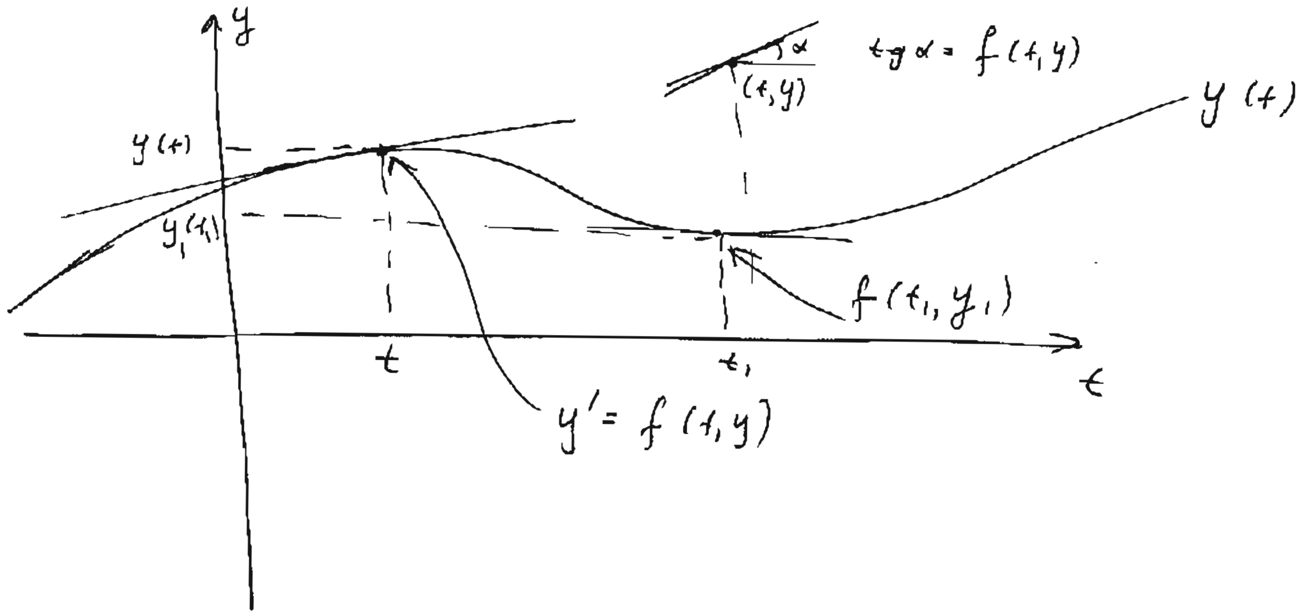
Initial value problem:

$$y(t) = \int_{t_0}^{t} f(t)\,dt + y_0$$ is a solution of (**), such that $y(t_0) = y_0$.

First order ODE.

$$y' = f(t, y), \quad y = y(t)$$

Geometrical interpretation

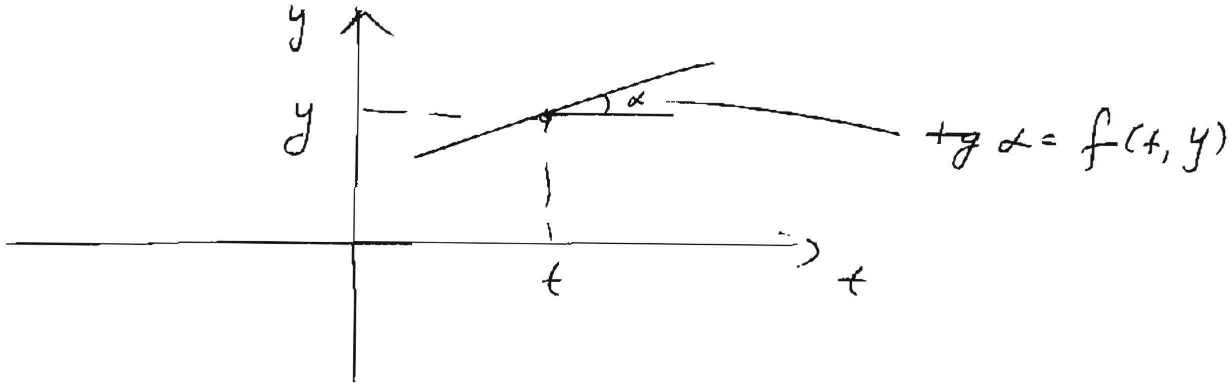


Def

Assume that at each point of the plane $\mathbb{R}^2_{(t,y)}$ a straight line passing through this point has been chosen. In this case we say that a direction field has been defined.

Remark

A diff. equation $y' = f(t, y)$ defines a direction field:



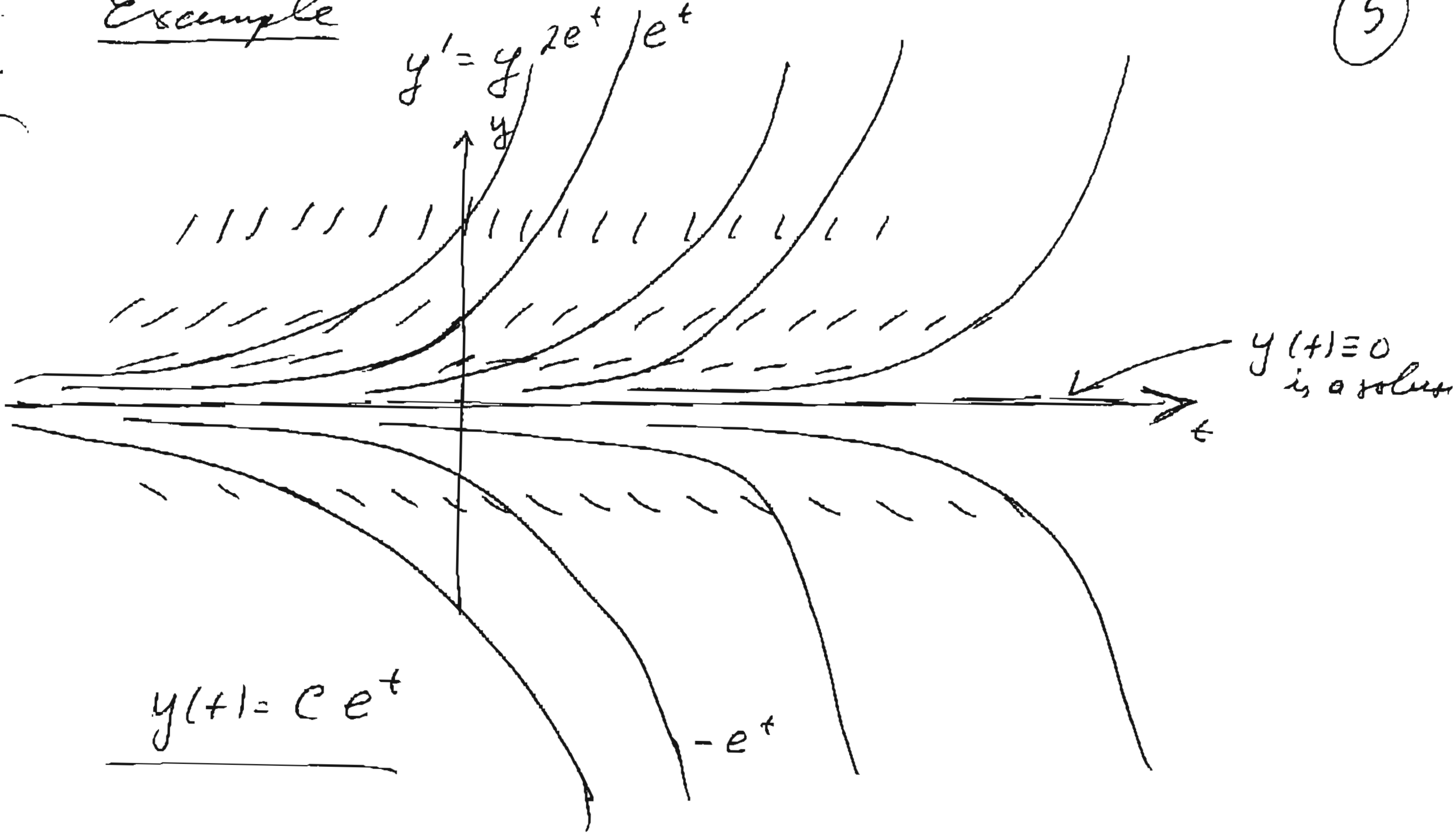
If $y(t)$ is a solution then the graph of $y(t)$ is tangent to the chosen line in every point.

Def A ~~line~~ curve which is (at each of its points) tangent to a direction field is called an <u>integral curve</u> of the direction field.

Geometrical problem

Given a direction field, find integral curves (or integral curve passing through a given point).

Example



$y(t) = Ce^t$

How to solve $y' = y$?

Notice that $\frac{d}{dt} \ln|y| = \frac{y'}{y}$, so

$$\frac{d}{dt}(\ln|y|) = 1,$$

exp:
$$\ln|y| = t + C_1$$

$$|y| = e^{t+C_1} = \underbrace{e^{C_1}} \cdot e^t,$$

$$\text{so } y = \underbrace{\pm \underbrace{e^{C_1}}}_{C \in \mathbb{R}} \cdot e^t$$

$y = Ce^t$ - general solution.

Equation of normal reproduction

$$y' = ky \quad , k > 0$$

$$\frac{d}{dt}\left(\ln|y|\right) = k$$

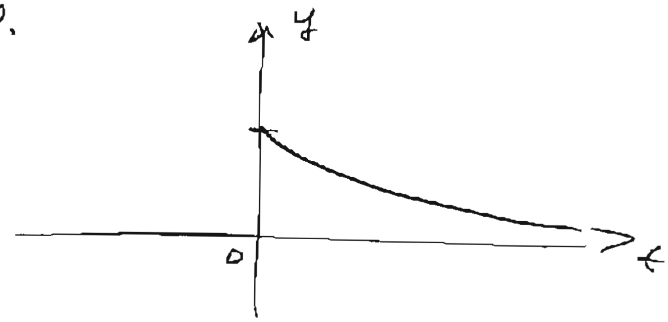
$$\ln|y| = kt + c_1$$

$$\underline{y = c\,e^{kt}} \quad \text{- general solution.}$$

Equation of radioactive decay

$$y' = -ky \quad , k > 0.$$

$$\underline{y = c\,e^{-kt}}$$

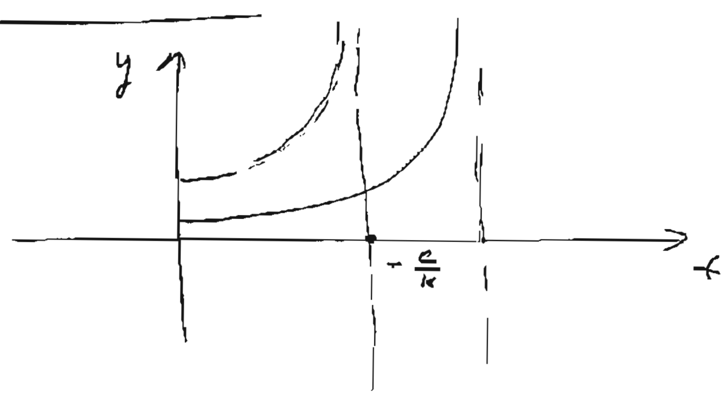


Explosion Equation

$$y' = ky^2 \qquad\qquad \frac{y'}{y^2} = \frac{d}{dt}\left(-\frac{1}{y}\right)$$

$$-\frac{1}{y} = kt + c$$

$$\underline{y = -\frac{1}{c + kt}} = \frac{1}{kt - (-c)}$$



First-order linear differential equations

Def

(nh) $\quad y' + a(t)\, y = b(t)$

is the first order linear differential equation

(h) $\quad y' + a(t)\, y = 0$ – homogeneous first order linear diff. equation

$b(t) \neq 0$ – nonhomogeneous.

How to solve $\quad y' + a(t)\, y = 0$

$$\frac{y'}{y} = -a(t)$$

$$\frac{d}{dt} \ln|y| = -a(t)$$

$$\ln|y| = -\int a(t)\,dt + C$$

$$\boxed{y(t) = C e^{-\int a(t)\,dt}}$$ – general solution of (h)

Example

$$y' = ty$$

$$\frac{y'}{y} = t$$

$$\boxed{y(t) = C e^{\frac{t^2}{2}}}$$

$$(\underbrace{Ce^{\frac{t^2}{2}}}_{y})' = \underbrace{Ce^{t^2/2}}_{y} \cdot t$$

$$y' = y \cdot t$$

How to solve $y' + a(t)y = b(t)$?

~~Method I~~ (Method of variation of constant)

$$y' + a(t)y = b(t) \qquad (nh)$$

Solve first $y' + a(t)y = 0$

$$y(t) = C e^{-\int a(t)dt}$$

Let us try to find a solution of (nh) in a form

$$y(t) = \varphi(t) \cdot e^{-\int a(t)dt}$$

$$\varphi'(t) e^{-\int a(t)dt} + \cancel{\varphi(t) \cdot \left(-a(t) e^{-\int a(t)dt}\right)} +$$

$$+ \cancel{a(t) \cdot \varphi(t) e^{-\int a(t)dt}} = b(t)$$

$$\cancel{\varphi'(t) = b(t) e^{\int a(t)dt}},$$

$$\cancel{\varphi(t) = \int b(t) e^{-\int a(t)dt} dt}$$

Find $\varphi(t) = G(t) + C$,

$$\underline{y(t) = (G(t) + C) e^{-\int a(t)dt}} \quad \text{- general solution.}$$

Example

$$y' + y + t = 0$$

Homog. equation:

$$y' = -y$$

$$y = Ce^{-t}$$

Let us try to find the solutions of the initial (non-homogeneous) equation in a form

$$y(t) = \varphi(t)e^{-t}$$

$$\varphi'(t)e^{-t} \underbrace{- \varphi(t)e^{-t} + \varphi(t)e^{-t}} + t = 0$$

$$\varphi'(t) = -te^{t}$$

$$\varphi(t) = -\int te^{t}dt + C =$$

$$= -(te^{t} - e^{t}) + C = C - te^{t} + e^{t}$$

$\underline{y(t) = (C - te^{t} + e^{t})e^{-t}}$ – general solution