Lecture 6

Existence and uniqueness theorem

Consider an initial value problem

$$(*)\quad \begin{cases} \dfrac{dy}{dt} = f(t,y) \\ y(t_0) = y_0 \end{cases}$$

1) $\exists$? solution?

2) Is it unique?

Thm (existence)

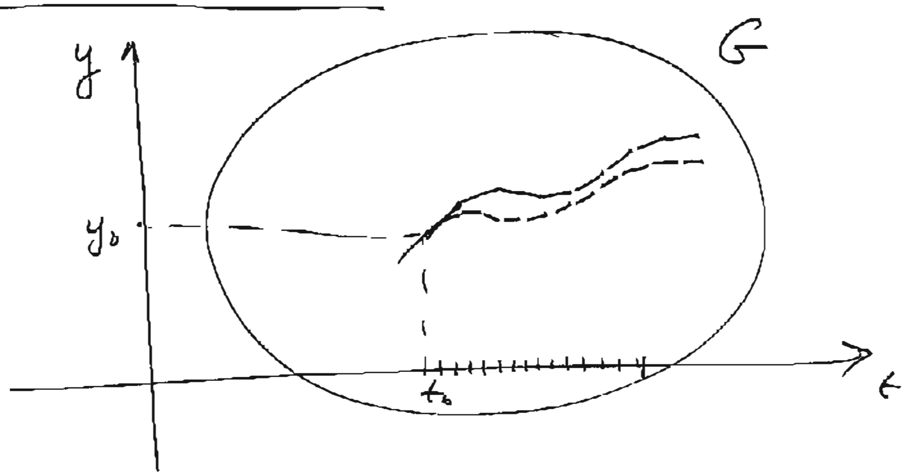
If $f \in C$ and bounded in some domain $G \subset \mathbb{R}^2$,

then $\forall (t_0, y_0) \in G$ locally $\exists$ a solution of $(*)$,

i.e. $\exists\ \alpha > 0$ s.t. $\exists\ y(t)$, $t \in [t_0, t_0+\alpha]$ s.t.

$y(t_0) = y_0$ and $y' = f(t,y)$ for $t \in (t_0, t_0+\alpha)$.

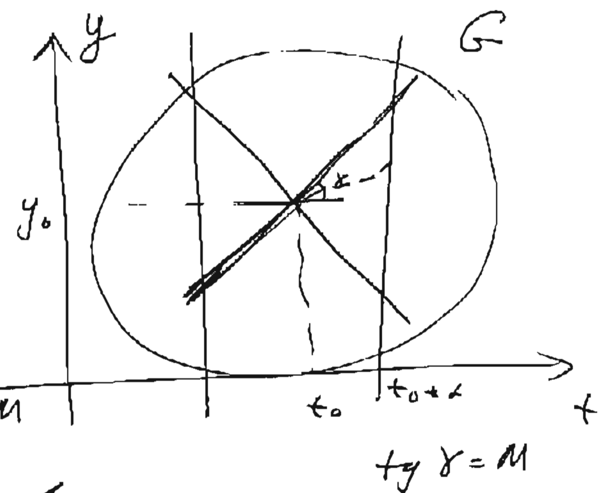
Euler lines



Take $t_0 < t_1 < t_2 < t_3 < \dots < t_n = t_0 + \alpha$,

and set the function φ to be linear on (t_i, t_{i+1}) and continuous on $(t_0, t_0+\alpha)$,

and $\varphi'(t) = f(t_i, \varphi(t_i))$ for $t \in (t_i, t_{i+1})$.



<u>Thm</u> (Peano)

If $|f| \le M$,

and α is such that

$\triangle$ formed by $\{t = t_0 + \alpha\}$,

and lines of slope M and $-M$

through (t_0, y_0) is in G then

the sequence of Euler approximations

have a subsequence convergent to

the solution of $(*)$ on $(t_0, t_0+\alpha]$.

<u>Picard approximations</u>

$$\frac{dy}{dt} = f(t, y)$$

$$\int_{t_0}^{t} \frac{dy(s)}{ds}\, ds = \int_{t_0}^{t} f(s, y(s))\, ds$$

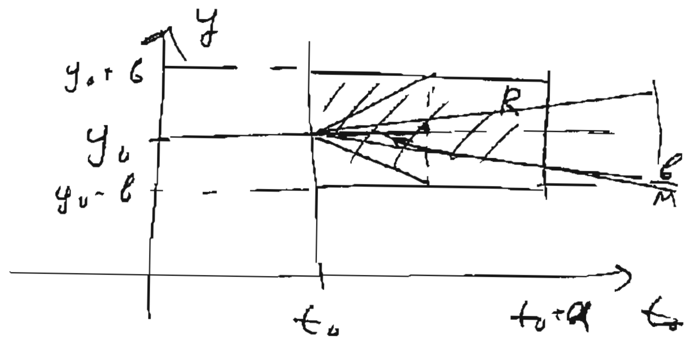
$$y(t) = y(t_0) + \int_{t_0}^{t} f(s, y(s))\, ds$$

Theorem

Consider (*), let f and $\frac{\partial f}{\partial y}$ be continuous in the rectangle

$$R = \{ (t, y) \mid t_0 \le t \le t_0 + a,\ |y - y_0| \le b \} \subset \mathbb{R}^2$$

Set



$$M = \max_{(t,y) \in R} |f(t,y)| ,$$

$$\alpha = \min \left(a, \frac{b}{M} \right)$$

Then (*) has a unique solution $y(t)$

on the interval $t_0 \le t \le t_0 + \alpha$.

($\Leftrightarrow$ If $y(t)$ and $z(t)$ are solutions of (*),

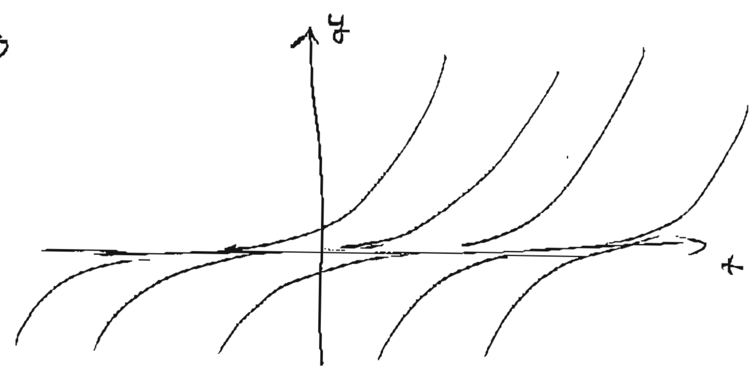
then $y(t) = z(t)$ for $t \in [t_0, t_0 + \alpha]$.

Example

$y' = 2\sqrt{|y|}$, $y(0) = 0$

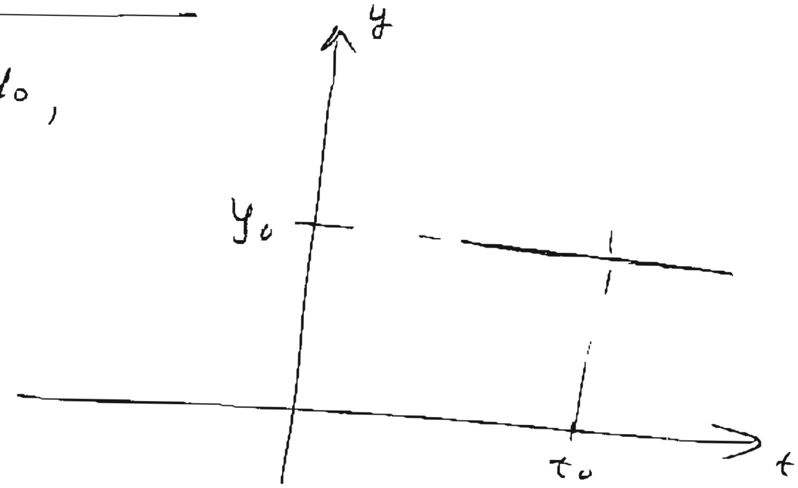
$y \equiv 0$ is a solution

$y(t) = t^2$ is also a solution!



Picard approximations

Take $y_0(t) = y_0$,



take

$$y_1(t) = y_0 + \int_{t_0}^{t} f(s, y_0(s))\,ds$$

$$y_2(t) = y_0 + \int_{t_0}^{t} f(s, y_1(s))\,ds$$

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s))\,ds$$

Then the sequence of functions will converge to a solution of $(*)$!

Example

$$\begin{cases} y' = ty \\ y(0) = 1 \end{cases}, \quad y_0(t) \equiv 1$$

$$y_1(t) = 1 + \int_0^t s\,ds = 1 + \frac{t^2}{2}$$

$$y_2(t) = 1 + \int_0^t \left(1 + \frac{s^2}{2}\right) s\,ds = 1 + \frac{s^2}{2} + \frac{s^4}{8}$$

$$y_3(t) = 1 + \int_0^t \left(1 + \frac{s^2}{2} + \frac{s^4}{8}\right) s\,ds = 1 + \frac{t^2}{2} + \frac{t^4}{8} + \frac{t^6}{48}$$

$y(t) = e^{t^2/2}$ is the solution.

On different definitions of exponent

Def 1

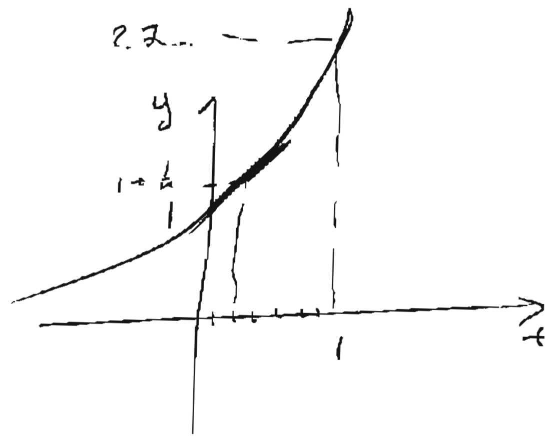
$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Def 2

$$e = 1+1+\frac{1}{2}+\frac{1}{3!}+\frac{1}{4!}+\dots = \sum_{k=0}^{\infty}\frac{1}{k!}$$

Def 3

Let $y(t)$ be the solution of the initial value problem



$$\begin{cases} y' = y \\ y(0) = 1 \end{cases}$$

Then $e = y(1)$.

Euler lines

$$0 = t_0 < t_1 < t_2 < \dots < t_n = 1, \quad t_k = \frac{1}{k}$$

$$\varphi(0) = 1, \quad \varphi(t_1) = 1 + \frac{1}{n},$$

$$\varphi(t_2) = 1 + \frac{1}{n} + \left(1+\frac{1}{n}\right)\cdot\frac{1}{n} = \left(1+\frac{1}{n}\right)^2$$

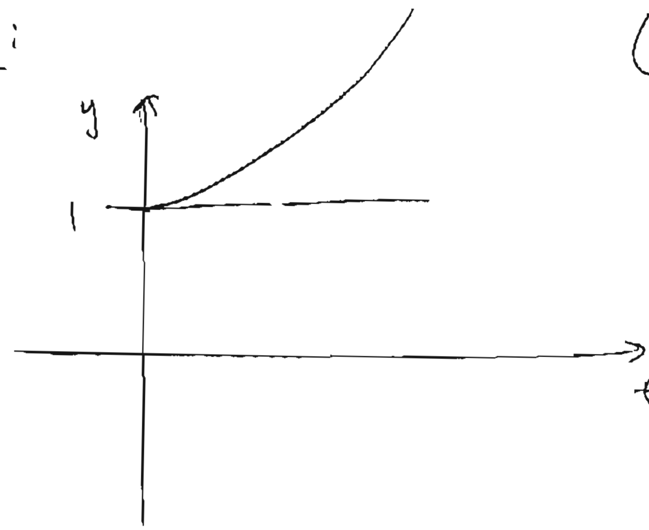
$$\varphi(t_3) = \left(1+\frac{1}{n}\right)^2 + \left(1+\frac{1}{n}\right)^2\cdot\frac{1}{n} = \left(1+\frac{1}{n}\right)^3$$

$$\vdots$$

$$\varphi(t_n) = \varphi(1) = \left(1+\frac{1}{n}\right)^n,$$

so $\left(1+\frac{1}{n}\right)^n \to e$ as $n\to\infty$.

Picard approximations:



$$\begin{cases} y' = y \\ y(0) = 1 \end{cases}$$

$$y_0(t) \equiv 1$$

$$y(t) = 1 + \int_0^t y(s)\,ds$$

$$y_{n+1}(t) = 1 + \int_0^t y_n(s)\,ds$$

$$y_1(t) = 1 + \int_0^t ds = 1 + t$$

$$y_2(t) = 1 + \int_0^t (1+s)\,ds = 1 + t + \frac{t^2}{2}$$

$$y_3(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6}$$

$$\cdots$$

$$y(t) = \lim_{n\to\infty} y_n(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots,$$

$$y(1) = e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots$$

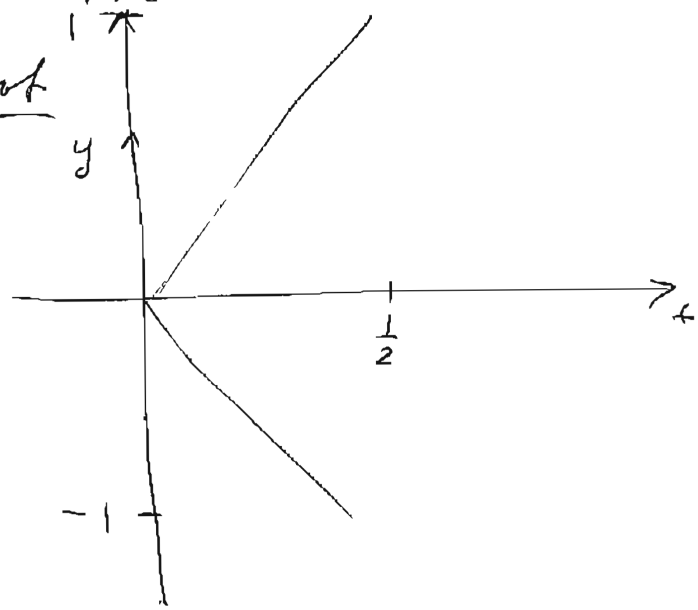
Example

Prove that the solution

$$\begin{cases} y' = e^{-t^2} + y^2 \\ y(0) = 0 \end{cases}$$

exists on the interval $0 \le t \le \frac{1}{2}$

Proof



Consider a rectangle

$$R = \{ t \in [0, \tfrac{1}{2}],\ |y| \le 1 \},$$

then $M = \max\limits_{(t,y)\in R} (e^{-t^2} + y^2) \le 2$,

and $\alpha = \min\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}$,

so solution exists on

the interval $[0, \alpha] = [0, \frac{1}{2}]$.