Introduction to differential equations

Exact equations

Def

$$M(t,y) + N(t,y)\frac{dy}{dt} = 0 \quad \text{is exact if}$$

$\exists\ \varphi(t,y)$ s.t.

$$M(t,y) = \frac{\partial \varphi}{\partial t} \text{ and } N(t,y) = \frac{\partial \varphi}{\partial y}.$$

In this case the equation has the form

$$\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial y}\cdot\frac{dy}{dt} = 0 \iff \frac{d}{dt}\left(\varphi(t,y)\right) = 0,$$

so $\underline{\varphi(t,y) = C}$ is a general solution.

Thm

$$M(t,y) + N(t,y)\frac{dy}{dt} = 0$$

is exact if, and only if

$$\underline{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}}.$$

Example

Any separable equation is exact:

$$\frac{dy}{dt} = \frac{g(t)}{f(y)}$$

$$g(t) - f(y)\frac{dy}{dt} = 0$$

$$M(t,y) = g(t),\ N(t,y) = -f(y), \quad \frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial t}$$

$$\varphi(t,y) = \int g(t)\,dt - \int f(y)\,dy,$$

$\int g(t)\,dt - \int f(y)\,dy = C$ – general solution of a separable equation.

$$M(t,y) + N(t,y)\frac{dy}{dt} = 0 \qquad (*)$$

Def A function $\mu(t,y)$ is called an integrating factor for $(*)$ if the equation

$$\mu(t,y)\cdot M(t,y) + \mu(t,y)\cdot N(t,y)\frac{dy}{dt} = 0$$

is exact.

Remark 1

$\mu(t,y)$ is an integrating factor $\iff$

$$\frac{\partial(\mu(t,y)M(t,y))}{\partial y} = \frac{\partial(\mu(t,y)N(t,y))}{\partial \cancel{y}\, t}, \text{ i.e.}$$

$$\boxed{\frac{\partial \mu}{\partial y}\cdot M(t,y) + \mu\cdot\frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial t}\cdot N(t,y) + \mu\cdot\frac{\partial N}{\partial t}}$$

Remark 2

Locally in a neighborhood of a non-degenerate point an integrating factor exists.

Example

$$\frac{dy}{dt} + a(t)y = b(t)$$

Let us try to find an integrating factor

$$(a(t)y - b(t)) + \frac{dy}{dt} = 0$$

$$M(t,y) = a(t)y - b(t)$$

$$N(t,y) = 1$$

$$\cancel{\mu(t,y)}$$

Let us try to find an integrating factor of the form $\mu = \mu(t)$.

$$\underbrace{\mu(t)\,(a(t)y - b(t))}_{M_1} + \underbrace{\mu(t)}_{N_1}\frac{dy}{dt} = 0$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial t}$$

$$\mu(t)a(t) = \frac{d\mu}{dt}, \quad \underline{\mu(t) = \int a(t)\,dt}$$

$$\begin{cases} \mu(t)\,(a(t)y - b(t)) = \dfrac{\partial \varphi}{\partial t} \\ \mu(t) = \dfrac{\partial \varphi}{\partial y} \quad \Rightarrow \varphi(t,y) = \mu(t)\cdot y + h(t) \end{cases}$$

$$\frac{\partial \varphi}{\partial t} = \frac{\partial \mu}{\partial t} \cdot y + h'(t) =$$

$$= \mu(t)\, a(t) y + h'(t) =$$

$$= \mu(t)\left(a(t) y - b(t)\right),$$

so $h' = -\mu(t) b(t)$, so

$$h(t) = -\int \mu(t) b(t)\, dt \cancel{+ C},$$

and

$$\varphi(t,y) = \mu(t)\, y - \int \mu(t) b(t)\, dt$$

General solution:

$\varphi(t,y) = C$, so

$\mu(t)\, y - \int \mu(t) b(t)\, dt = C$, or

$$y = (\mu(t))^{-1} \cdot \left(C + \int \mu(t) b(t)\, dt\right),$$

where $\mu(t) = \int a(t) dt$.

Homogeneous Equations

Def An equation

$$\frac{dy}{dt} = f\left(\frac{y}{t}\right)$$

is called homogeneous differential equation.

Solution

Consider a function $u(t) = \frac{y}{t}$,

then $y(t) = u(t) \cdot t$, and

$y' = u(t) + u' \cdot t$, so

$u(t) + u'(t) \cdot t = f(u)$, and

$$\boxed{u' = \frac{f(u) - u}{t}}$$ separable equation,

$$\int \frac{du}{f(u)-u} = \int \frac{dt}{t},$$

$$\ln |t| = \int \frac{du}{f(u)-u} + C$$

Example

$$y' = \frac{y - t}{y + t}$$

$$y' = \frac{y/t - 1}{y/t + 1} \quad \text{— homogeneous!}$$

$$u = \frac{y}{t}, \quad y = u \cdot t, \quad y' = u't + u$$

$$u' \cdot t + u = \frac{u-1}{u+1}$$

$$u' \cdot t = \frac{u-1}{u+1} - u = \frac{u - 1 - u^2 - u}{u+1} = -\frac{1+u^2}{u+1}$$

$$\int \frac{(1+u)\,du}{1+u^2} = -\int \frac{dt}{t}$$

$$\operatorname{arctg} u + \frac{1}{2} \ln(1+u^2) = -\ln|t| + C/2$$

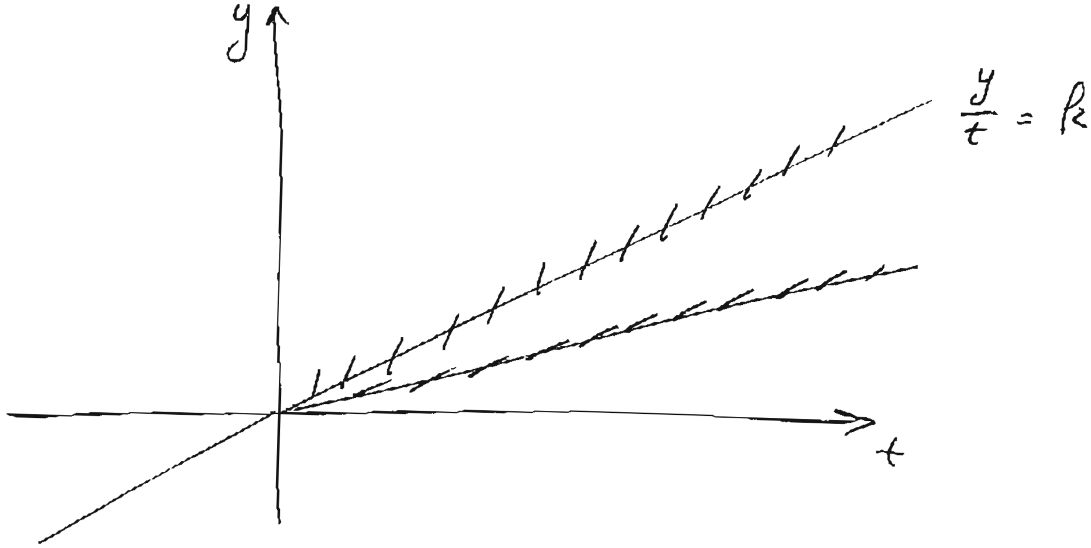
$$2 \operatorname{arctg} u + \ln(1+u^2) + \ln(t^2) = C$$

$$u = y/t$$

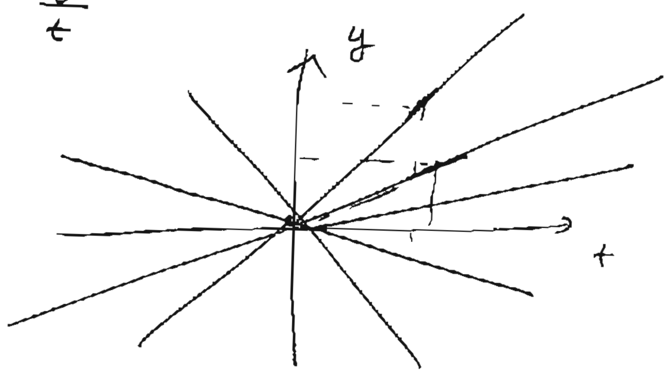
$$\boxed{2 \operatorname{arctg} \frac{y}{t} + \ln(t^2 + u^2) = C}$$

general solution

Geometrical interpretation



$$\frac{dy}{dt} = \frac{y}{t}$$

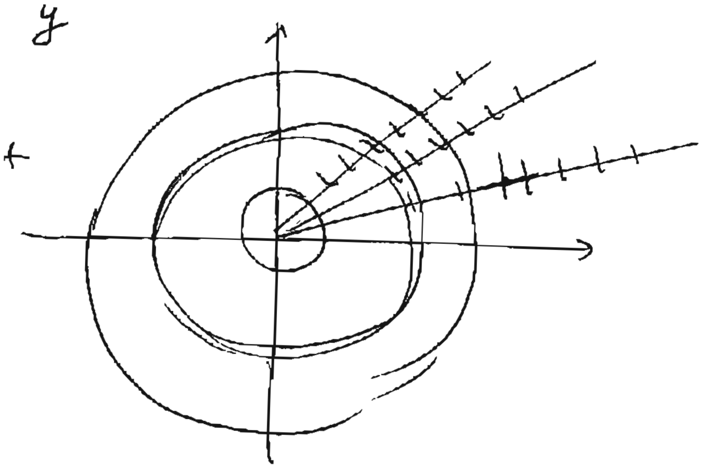


$$\int \frac{dy}{y} = \int \frac{dt}{t}$$

$$\ln|y| = \ln|t| + \tilde{c}$$

$$\underline{y = c\,t}, \quad c \in \mathbb{R}$$

$$\frac{dy}{dt} = -\frac{t}{y}$$



$$\int y\,dy = -\int t\,dt$$

$$\frac{y^2}{2} = -\frac{t^2}{2} + c$$

$$\boxed{y^2 + t^2 = 2c}$$