Introduction to differential equations

Exact equations

$$\frac{d}{dt}\left(\varphi(y,t)\right)=0$$

general solution $\boxed{\varphi(y,t)=C}$

$$\frac{d}{dt}\left(\varphi(y,t)\right)=\boxed{\frac{\partial\varphi}{\partial y}\cdot\frac{dy}{dt}+\frac{\partial\varphi}{\partial t}=0}$$

Def If $\exists\varphi$ s.t. the equation has this form then the equation is exact

Example

$$y'\cdot(2y\sin t+1)+y^2\cos t=0$$

$$\frac{d}{dt}\left(y^2\sin t+y\right)=0$$

$\boxed{y^2\sin t+y=C}$ general solution

Example

$$y^2+2y^3t+\cos t+(2yt+3y^2t-\sin y)\,y'=0$$

$$\frac{d}{dt}\left(\sin t+y^2t+y^3t+\cos y\right)=0$$

1) How to find out that the equation is exact?

2) How to find the function φ?
(and therefore to solve the equation)?

Theorem

The equation

$$M(t,y) + N(t,y)\frac{dy}{dt} = 0$$

is exact (i.e. $\exists\, \varphi$ such that $M(t,y) = \frac{\partial \varphi}{\partial t}$ and $N(t,y) = \frac{\partial \varphi}{\partial y}$)

if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$.

Proposition.

Let $M(t,y)$ and $N(t,y)$ be C^1-functions (i.e. M, N are continuous, differentiable, and their partial derivatives are continuous), and defined in $R = \{(t,y) \mid a < t < b,\ c < y < d\}$.

There exists $\varphi(t,y)$ such that

$$M(t,y) = \frac{\partial \varphi}{\partial t} \text{ and } N(t,y) = \frac{\partial \varphi}{\partial y}$$

if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t} \text{ in } R.$$

Proof

If $M(t,y) = \frac{\partial \varphi}{\partial t}$ then

$$\varphi(t,y) = \int M(t,y)\,dt + h(y) \text{ for some function } h(y).$$

Then

$$\frac{\partial \varphi}{\partial y} = \int \frac{\partial M(t,y)}{\partial y}\,dt + h'(y)$$

Therefore $\frac{\partial \varphi}{\partial y} = N(t,y)$ if, and only if

$$N(t,y) = \int \frac{\partial M(t,y)}{\partial y}\,dt + h'(y), \text{ or}$$

$$h'(y) = \underbrace{N(t,y) - \int \frac{\partial M(t,y)}{\partial y}\,dt}$$

Therefore this function is independent of t,

and

$$\frac{\partial}{\partial t}\left(N(t,y) - \int \frac{\partial M(t,y)}{\partial y}\,dt\right) =$$

$$= \boxed{\frac{\partial N}{\partial t} - \frac{\partial M}{\partial y} = 0}$$

Therefore if $\frac{\partial \varphi}{\partial y} = M$ and $\frac{\partial \varphi}{\partial t} = N$ for some $\varphi(t,y)$

then $\frac{\partial N}{\partial t} = \frac{\partial M}{\partial y}$.

If $\frac{\partial N}{\partial t} = \frac{\partial M}{\partial y}$ then

$$N(t,y) - \int \frac{\partial M(t,y)}{\partial y}\,dt \quad \text{depends on } y \text{ only,}$$

so we can take

$$h(y) = \int \left[N(t,y) - \int \frac{\partial M(t,y)}{\partial y}\,dt \right] dy,$$

and if

$$\varphi(t,y) = \int M(t,y)\,dt + \int \left[N(t,y) - \int \frac{\partial M(t,y)}{\partial y}\,dt \right] dy, \text{ then}$$

$$M(t,y) = \frac{\partial \varphi}{\partial t} \text{ and } N(t,y) = \frac{\partial \varphi}{\partial y} \qquad \square$$

<u>Example</u>

$$\underline{e^{-y} - (2y + te^{-y})\frac{dy}{dt} = 0}$$

$$M(t,y) = e^{-y}$$
$$N(t,y) = -(2y + te^{-y})$$

$$\frac{\partial M}{\partial y} = -e^{-y} = \frac{\partial N}{\partial t},$$ so this is an exact equation, so

$\exists\ \varphi(t,y)$ such that

$$\begin{cases} \frac{\partial \varphi}{\partial t} = e^{-y} \\ \frac{\partial \varphi}{\partial y} = -(2y + te^{-y}) \end{cases}$$

Therefore

$$\varphi(t,y) = t e^{-y} + h(y), \text{ so}$$

$$\frac{\partial \varphi}{\partial y} = \underbrace{-t e^{-y}} + h'(y) = -2y \underbrace{-t e^{-y}}$$

$$\text{so } h'(y) = -2y,$$

$$\text{so } \underline{h(y) = -y^2},$$

and $\varphi(t,y) = t e^{-y} - y^2$,

the initial equation has the form

$$\frac{d}{dt}\left(t e^{-y} - y^2\right) = 0, \text{ and}$$

has a general solution

$$\boxed{t e^{-y} - y^2 = C}$$

<u>Def</u> $$\underline{M(t,y) + N(t,y)\frac{dy}{dt} = 0} \quad (*)$$

~~If~~ A function $\mu(t,y)$ is called an <u>integrating factor</u> for $(*)$ if the equation

$$\mu(t,y)\, M(t,y) + \mu(t,y)\, N(t,y)\frac{dy}{dt} = 0$$

is exact.

Proposition 2

$\mu(t,y)$ is an integrating factor of $(*)$ if

$$M\frac{\partial\mu}{\partial y}+\mu\frac{\partial M}{\partial y}=N\frac{\partial\mu}{\partial t}+\mu\frac{\partial N}{\partial t}$$

Proof

$$\underbrace{\mu(t,y)M(t,y)}_{M_1}+\underbrace{\mu(t,y)N(t,y)}_{N_1}\frac{dy}{dt}=0$$

$$\frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial t}$$

$$M\frac{\partial\mu}{\partial y}+\mu\frac{\partial M}{\partial y}=N\frac{\partial\mu}{\partial t}+\mu\frac{\partial N}{\partial t}\qquad\square$$

Example

Assume that $y_1(t)$ and $y_2(t)$ are two different solutions of a differential equation

$$y'+a(t)y=b(t)\qquad (*)$$

Find a general solution.

Solution:

$$y_1'+a(t)y_1=b(t)$$
$$y_2'+a(t)y_2=b(t)$$

$$(y_1-y_2)'+a(t)(y_1-y_2)=0,\quad\text{and } y_1-y_2\not\equiv 0.$$

Therefore a general solution of the homogeneous equation has a form $C\cdot(y_1(t)-y_2(t))$, and a general solution of $(*)$ has a form $y_1+C\cdot(y_1-y_2)$, $C\in\mathbb{R}$

Example

$$\frac{dy}{dt} + a(t)y = b(t)$$

Let us try to find an integrating factor

$$\underbrace{\mu(t,y)}_{N_1(t,y)} \frac{dy}{dt} + \underbrace{\left(a(t)y - b(t)\right)\mu(t,y)}_{M_1(t,y)} = 0$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial t}$$

$$\frac{\partial \mu}{\partial t} = a(t)\mu(t,y) + a(t)\frac{\partial \mu(t,y)}{\partial y} y - b(t)\frac{\partial \mu(t,y)}{\partial y}.$$

Assume that $\mu(t,y) = \mu(t)$ (i.e. μ does not depend on y)

Then $\frac{\partial \mu}{\partial t} = a(t)\mu(t) \Rightarrow \mu(t) = \exp\left(\int a(t)\,dt\right)$.

$$N_1(t,y) = \mu(t) = \frac{\partial \varphi}{\partial y} \Rightarrow \underline{\varphi(t,y) = \mu(t)y + k(t)},$$

$$M_1(t,y) = \left(a(t)y - b(t)\right)\mu(t) = \frac{\partial \varphi}{\partial t}$$

so $\left(a(t)y - b(t)\right)\mu(t) = \underbrace{\frac{d\mu}{dt}}_{\mu(t)\cdot a(t)} y + \frac{dk}{dt}$

$$k'(t) = -\mu(t)b(t), \text{ so}$$

general solution: $\boxed{\mu(t)y - \int \mu(t)b(t)\,dt = C, \text{ where } \mu(t) = \exp\left(\int a(t)\,dt\right)}$